Summary of last lectures

- singular limit

$$\lim_{h\to 0}\ \lim_{N\to\infty} \langle \hat{O} \rangle \neq 0$$ in symmetry broken phase

(order parameter: $\hat{O}$)

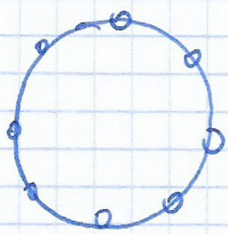
$[\hat{H}, \hat{U}] = 0$ ($\hat{U}$: symmetry)

$\hat{H}|\Psi\rangle = E|\Psi\rangle$ — SSB

but $\hat{U}|\Psi\rangle \neq |\Psi\rangle$

not symmetric

- harmonic crystal



$$\hat{H} = \sum_j \frac{p_j^2}{2m} + \frac{\kappa}{2}\sum_j (x_j - x_{j+1})^2 =$$

$$= H_{k=0} + \underbrace{\sum_{k\neq 0} \varepsilon_k B_k^\dagger B_k}_{} + \text{const.}$$

phonons, quantized sound waves of a crystal → Goldstone modes

$$\hat{H}_{k=0} = \frac{\hat{P}_{tot}^2}{2Nm}$$

$\hat{P}_{tot} = \sum_j p_j$ – total momentum

$[\hat{H}, \hat{P}_{tot}] = 0$ → all eigenstates are $\Psi_{P_{tot}} = e^{\frac{i P_{tot} x}{\hbar}}$

with $E_{P_{tot}} = \dfrac{P_{tot}^2}{2mN}$

excited

$\Delta E \sim \dfrac{1}{N}$

ground

tower of states

ground state $\Psi_{P_{tot}=0} = \dfrac{1}{\sqrt{N}}$

In the thermodynamic limit $N\to\infty$ the excited states $P_{tot}\neq 0$ are degenerate with $P_{tot}=0$ state.

<u>One can form a stable (not dispersing) wave packet which is localized</u>

In the thermodynamic limit $N \to \infty$ all states in the $H_{k=0}$ Hamiltonian are degenerate with the ground state.

Therefore, a wave packet of total momentum states with a well defined center-of-mass position would have the same energy expectation value as the zero-total-momentum eigenstate.

These states do not contribute to the thermodynamics

$$Z_{k=0} = \sum_{P_{tot}} e^{-\beta H_{k=0}} = \int_{-\infty}^{\infty} dp \, e^{-\beta \frac{p^2}{2mN}} = \left\{ \sqrt{\frac{\beta}{2mN}}\, p = x \right\}$$

$$Z_{k=0} \sim \sqrt{N}$$

$$F_{k=0} = -k_B T \ln Z_{k=0} \sim \ln \sqrt{N}$$

$$\frac{F_{k=0}}{F_{k\neq 0}} \sim \frac{\ln N}{N} \xrightarrow[N\to\infty]{} 0$$

Therefore, this part of the spectrum is called the <u>thin spectrum</u> of the quantum crystal.

To see how the thin spectrum conspires to break the symmetry we add a small perturbation

$$\hat{H}_{k=0} = \frac{\hat{P}_{tot}^2}{2mN} + \mu \, \hat{X}_{CM}^2$$

$X_{CM} = \frac{1}{N}\sum_j x_j$ – center of mass position

This is a Harmonic oscillator problem

$\omega = \sqrt{\frac{2\mu}{mN}}$, $E_n = \hbar\omega\left(n + \frac{1}{2}\right)$

$$\Psi_0(x) = \left(\frac{2mN\mu}{\pi^2\hbar^2}\right)^{1/8} e^{-\sqrt{\frac{mN\mu}{2\hbar^2}}\, x^2}$$

↑ Gaussian wave packet

$$\sigma^2 = \frac{\hbar}{\sqrt{2mN\mu}}$$

The SSB occurs by applying two non-commuting limits

$$\lim_{N\to\infty} \lim_{\mu\to 0} |\Psi_0(x)|^2 = \text{const.} \to 0$$

$$\lim_{\mu\to 0} \lim_{N\to\infty} |\Psi_0(x)|^2 = \delta(x)$$

The thermodynamic limit is singular!

In both cases $E_0 = \frac{\hbar\omega}{2} = 0$

> In the thermodynamic limit the symmetry broken, localized state of the crystal has the same energy as the exact, plane-wave ground state.

Estimate: iron (Fe) $m = 9.27 \cdot 10^{-26}$ kg, $a = 2.856 \cdot 10^{-10}$ m,

$N = 4 \cdot 10^{22}$ atoms, $\mu \sim 10^{-19}$ N/m $\rightarrow \boxed{\sigma \sim 2 \cdot 10^{-12} \text{ m}}$

This simple model illustrates a generic mechanism for SSB:

- due to the thermodynamic limit there are singular (non commuting) limits

- there is an order parameter $\hat{O}$ whose eigenstates are inequivalent and different

- $[\hat{O}, \hat{H}] \neq 0 \rightarrow$ eigenstates of $\hat{O}$ are not eigenstates of $\hat{H}$
- however $\langle [\hat{O}, \hat{H}] \rangle_{N\to\infty} = 0$ and

  eigenstates of $\hat{O}$ are degenerate with eigenstates of $\hat{H}$
- this leads to a stable ground or thermal equilibrium SSB state.

- excited states are known as Goldstone modes / quasiparticles.

§4 Ferromagnet – a prominent exception

Heisenberg exchange

$$\boxed{\hat{H} = -J \sum_{\langle ij \rangle} \bar{S}_i \cdot \bar{S}_j} \qquad J > 0$$

The total spin is

$$\boxed{\bar{S} = \sum_i \bar{s}_i}$$

since $[\bar{S}, \hat{H}] = 0$ therefore the system is rotationally invariant $\boxed{\hat{U}(\phi) = e^{-\frac{i}{\hbar}\phi \cdot \bar{S}}}$

$$\boxed{[\hat{U}(\phi), \hat{H}] = 0}$$

If we write the Hamiltonian in the form

$$H = -J \sum_{\langle ij \rangle} \left(s_i^z s_j^z + \frac{1}{2}(s_i^+ s_j^- + s_i^- s_j^+) \right)$$

we easy can see that the state with all spins up / down

$$|\ldots \uparrow\uparrow\uparrow \ldots\rangle \quad / \quad |\ldots \downarrow\downarrow\downarrow \ldots\rangle$$

is a eigenstate of $\hat{H}$. It is a ground state:

$$\boxed{\hat{H} |\ldots \uparrow\uparrow\uparrow \ldots\rangle = -JN\left(\frac{\hbar}{2}\right)^2 |\ldots \uparrow\uparrow\uparrow \ldots\rangle}$$

The ground state has lower symmetry then the Hamiltonian

To understand this mathematically we define an orientation of the total spin vector

$$\boxed{\hat{n} := \frac{\hat{S}}{N/2}}$$

and find the commutation relation

$\alpha\beta = x, y, z$

$$[\hat{n}_\alpha, \hat{n}_\beta] = \frac{1}{(\frac{N}{2})^2}[\hat{S}_\alpha, \hat{S}_\beta] = \frac{i\hbar}{(\frac{N}{2})^2}\varepsilon^{\alpha\beta\gamma}\hat{S}_\gamma =$$

$$= \frac{i\hbar}{\frac{N}{2}}\varepsilon^{\alpha\beta\gamma}\hat{n}_\gamma \xrightarrow[N\to\infty]{\hbar\to 0} 0$$

the orientation vector becomes classical in the thermodynamic limit. A state described by $\bar{n}$ is regarded as a classical object.

In fact, there are infinitely many such orientations. Indeed, using

$$e^{\frac{i}{\hbar}\bar{\Phi}\cdot\bar{S}} = e^{\frac{i}{2}\bar{\Phi}\cdot\bar{\sigma}} = \cos\left(\frac{\Phi}{2}\right) + i\bar{\sigma}\frac{\bar{\Phi}}{|\bar{\Phi}|}\sin\left(\frac{\Phi}{2}\right)$$

we find

$$e^{\frac{i}{2}\bar{\Phi}\cdot\bar{\sigma}}|\uparrow\rangle = \cos\left(\frac{\Phi}{2}\right) + i\sin\left(\frac{\Phi}{2}\right)\left(\frac{\Phi_z}{|\Phi|}|\uparrow\rangle + \frac{\Phi_x + i\Phi_y}{|\Phi|}|\downarrow\rangle\right).$$

Therefore we check the overlap of different oriented states

$$\langle\cdots\uparrow\uparrow\cdots|\hat{U}(\Phi)|\cdots\uparrow\uparrow\cdots\rangle = \lim_{N\to\infty}\left(\cos\left(\frac{\Phi}{2}\right) + i\frac{\Phi_z}{|\Phi|}\sin\left(\frac{\Phi}{2}\right)\right)^N$$

unless $\bar{\Phi} \parallel z$

$$\left|\cos\frac{\Phi}{2} + i\frac{\Phi_z}{\Phi}\sin\frac{\Phi}{2}\right|^2 = 1 - \frac{\Phi_x^2 + \Phi_y^2}{\Phi^2}\sin^2\left(\frac{\Phi}{2}\right) < 1$$

and the limit $N \to \infty$ gives zero.

There are infinitely many ground states oriented differently and orthogonal to each other.

However, since $[\hat{n}_\alpha, \hat{n}_\beta] = 0$

they are classical and do not superimpose. Only one is spontaneously selected (e.g. by Earth magnetic field) in a classical mechanism.

The low lying excited states are Goldstone modes $\rightarrow$ magnons / spinwaves with a dispersion

$$\varepsilon_k = J(1 - \cos(ka)) \simeq -\frac{J}{2} a^2 k^2$$

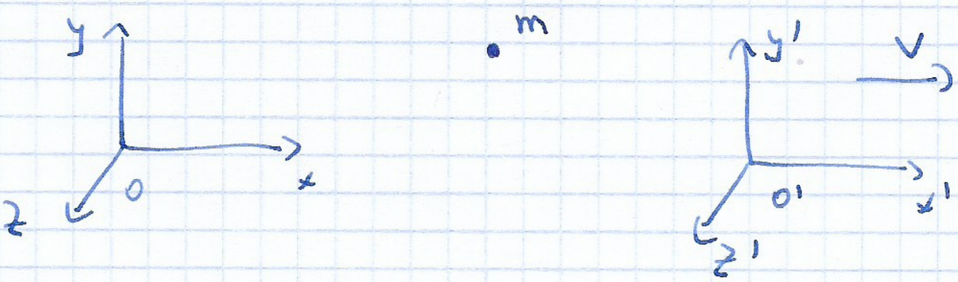
(not $\varepsilon_k \sim k$ like in phonons!)

(tutorial)

III GALILEAN AND GAUGE TRANSFORMATIONS IN QUANTUM MECHANICS

§1. Landé's - Lévy-Leblond's pseudoparadox

Am. J. Phys. **44**, 1130 (1976)



$$\boxed{\begin{aligned} x' &= x - vt \\ t' &= t \end{aligned}}$$

Galilean transformation

$$\boxed{p' = p - mv}$$

Let's check de Broglie and Einstein hypothesis

de Broglie

$$\boxed{p = \frac{h}{\lambda}}$$

Einstein

$$\boxed{E = h\nu}$$

Momentum transformation modifies de Broglie wave length

$$p' = \frac{h}{\lambda'}$$

$$p - mv = \frac{h}{\lambda'}$$

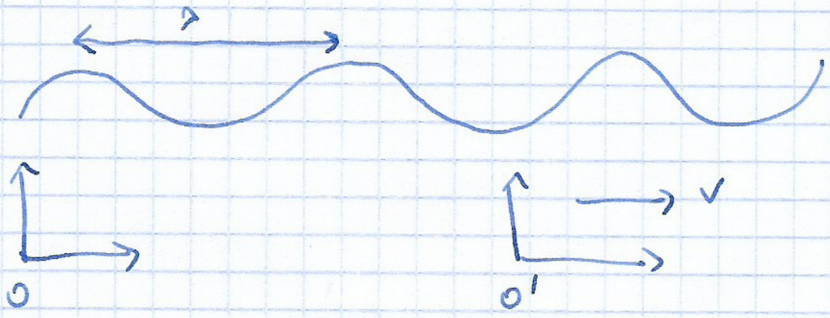
$$\frac{h}{\lambda} - mv = \frac{h}{\lambda'} \quad \rightarrow \quad \boxed{\frac{1}{\lambda'} = \frac{1}{\lambda} - \frac{mv}{h}}$$

quantum Doppler effect

In classical wave physics $\lambda' = \lambda$!

Indeed, for a plane wave

$$\psi(x,t) = A \sin\left[2\pi\left(\frac{x}{\lambda} - \nu t\right)\right] \quad \text{in } O$$



in O' $\qquad \psi'(x',t') = \psi(x,t)$

$$\psi'(x',t') = \psi(x,t) = A \sin\left[2\pi\left(\frac{x'+Vt'}{\lambda} - \nu t'\right)\right] =$$

$$= A \sin\left[2\pi\left(\frac{x'}{\lambda} - \left(\nu - \frac{V}{\lambda}\right)t'\right)\right]$$

therefore

$$\boxed{\begin{aligned} \lambda' &= \lambda \\ \nu' &= \nu - \frac{V}{\lambda} \end{aligned}}$$

nonrelativistic Doppler effect

Einstein relation:

$$E = \frac{p^2}{2m} \qquad E' = \frac{p'^2}{2m} = \frac{(p - mV)^2}{2m} = \frac{p^2}{2m} - Vp + \frac{mV^2}{2}$$

Einstein plus Doppler effect

$$E' = h\nu' = h\nu - \frac{hV}{\lambda} = h\nu - pV$$

§2. Galilean transformation

m - mass of the particle

Wigner operator (*) $\hat{u}(\bar{v},t)$ is defined

$$\boxed{\begin{aligned} \hat{u}^{+}(\bar{v},t)\,\hat{\bar{r}}\,\hat{u}(\bar{v},t) &= \hat{\bar{r}} - \bar{v}t \\ \hat{u}^{+}(\bar{v},t)\,\hat{\bar{p}}\,\hat{u}(\bar{v},t) &= \hat{\bar{p}} - m\bar{v} \end{aligned}} \qquad (*)$$

(*) Wigner theorem: for any transformation

$$G: \bar{R} \to \bar{R}' = G(\bar{R})$$

which conserves probability $|\langle\Psi|\Phi\rangle| = |\langle\Psi'|\Phi'\rangle|$
there is an operator $\hat{u} = \hat{u}_G$ such that

$$|\Psi'\rangle = \hat{u}\,|\Psi\rangle,$$

which is unitary or antiunitary. □

We assume that

$$\boxed{\hat{u}(\bar{v},t) = \hat{u}_p(\bar{v},t)\cdot\hat{u}_r(\bar{v},t)}$$

where $\hat{u}_p$ is a function of $\hat{\bar{p}}$ and $\hat{u}_r$ is a function of $\hat{\bar{r}}$.
If we assume

$$\hat{u}_p^{+}(\bar{v},t)\,\hat{\bar{r}}\,\hat{u}_p(\bar{v},t) = \hat{\bar{r}} - \bar{v}t$$

$$\hat{u}_r^{+}(\bar{v},t)\,\hat{\bar{p}}\,\hat{u}_r(\bar{v},t) = \hat{\bar{p}} - m\bar{v}$$

equations (*) are satisfied.

$\hat{U}_p(\bar{v},t)$ is a translation operator

$$\hat{U}_p(\bar{v},t) = e^{\,i\,\underset{\substack{\uparrow\\ \text{arbitrary}\\ \text{phase}}}{\gamma_p(\bar{v},t)} + i\,\underbrace{t\bar{v}}_{\bar{a}}\cdot\hat{\bar{p}}/\hbar}$$

$\hat{U}_r(\bar{v},t)$ – translates $\hat{\bar{p}}$ but leaves $\hat{\bar{r}}$ invariant.
It must have a form

$$\hat{U}_r(\bar{v},t) = e^{\,i\,\underset{\substack{\uparrow\\ \text{arbitrary}\\ \text{phase}}}{\gamma_r(\bar{v},t)} - i\,\underbrace{m\bar{v}}_{\bar{p}_0}\cdot\hat{\bar{r}}/\hbar}$$

using $e^{\hat{A}}e^{\hat{B}} = e^{\frac{1}{2}[\hat{A},\hat{B}]}e^{\hat{A}+\hat{B}}$
we get

$$\hat{U}(\bar{v},t) = e^{i\gamma_p(\bar{v},t) + i\frac{\bar{v}\cdot\hat{\bar{p}}t}{\hbar}}\, e^{i\gamma_r(\bar{v},t) - \frac{im\bar{v}\cdot\hat{\bar{r}}}{\hbar}} =$$

$$= e^{i(\gamma_p(\bar{v},t) + \gamma_r(\bar{v},t) - \frac{\frac{1}{2}m\bar{v}^2 t}{\hbar})}\, e^{i(t\hat{\bar{p}} - m\hat{\bar{r}})\cdot\bar{v}/\hbar}$$

$$[it\bar{v}\cdot\hat{\bar{p}}, -im\bar{v}\cdot\hat{\bar{r}}] = tm\bar{v}^2\underbrace{[\hat{\bar{p}},\hat{\bar{r}}]}_{=-i\hbar} = -tm\bar{v}^2 i\hbar$$

choosing $\quad \gamma_p(\bar{v},t) + \gamma_r(\bar{v},t) = \frac{1}{2}m\bar{v}^2 t + {}^{(*)}\underset{=0}{\gamma_0(\bar{v})}$

we get
$$\boxed{\hat{U}(\bar{v},t) = e^{i\frac{(\hat{\bar{p}}t - m\hat{\bar{r}})\cdot\bar{v}}{\hbar}}}$$

*) this is only compatible choice with Galilean covariance
e.g. for free particle $\hat{H} = \hat{\bar{p}}^2/2m$

$$[\hat{U},\hat{H}] + i\partial_t\hat{U} = 0 \rightarrow \frac{d\gamma(\bar{v},t)}{dt} = \frac{1}{2}m\bar{v}^2$$

$$\rightarrow \gamma(\bar{v},t) = \frac{1}{2}m\bar{v}^2 t + \gamma_0(\bar{v})$$

The wave function transforms as

$$\Psi'(\bar{r}', t) = \langle \bar{r}' | \hat{u}(\bar{v}, t) | \Psi(t) \rangle =$$

$$= \int d_d r'' \int \frac{d_d p}{(2\pi)^d} \langle \bar{r}' | e^{\frac{i t \bar{v}\cdot\hat{\bar{p}}}{\hbar}} | \bar{p} \rangle \langle \bar{p} | e^{-i\frac{m\bar{v}\cdot\hat{\bar{r}}}{\hbar}} | \bar{r}'' \rangle \langle \bar{r}'' | \Psi(t) \rangle$$

$$= \int d_d r'' \int \frac{d_d p}{(2\pi)^d} e^{i\gamma(\bar{v},t) - i m \frac{\bar{v}\cdot\bar{r}''}{\hbar} + i \bar{p}\frac{(\bar{v}\cdot t + \bar{r}' - \bar{r}'')}{\hbar}} \Psi(\bar{r}'', t)$$

$$= \int d r'' \, e^{i\gamma(\bar{v},t) - i m \frac{\bar{v}\cdot\bar{r}''}{\hbar}} \delta(\bar{v}\cdot t + \bar{r}' - \bar{r}'') \Psi(\bar{r}'', t) =$$

$$= e^{\frac{\frac{i}{2} m \bar{v}^2 t - i m \bar{v}\cdot\bar{r}}{\hbar}} \Psi(\bar{r}, t)$$

where $\bar{r} = \bar{r}' + \bar{v}\cdot t$.

<u>Galilean transformation</u>

$$\boxed{\begin{aligned} \hat{u}(\bar{v}, t) &= e^{\frac{i}{\hbar}(\hat{\bar{p}} t - m\hat{\bar{r}})\cdot\bar{v}} \\ \hat{\bar{r}}_T &= \hat{u}^+ \hat{\bar{r}} \hat{u} = \hat{\bar{r}} - \bar{v} t \\ \hat{\bar{p}}_T &= \hat{u}^+ \hat{\bar{p}} \hat{u} = \bar{p} - m\bar{v} \\ \Psi_T(\bar{r}', t') &= e^{\frac{i}{\hbar}\left(\frac{m v^2}{2} t - m\bar{v}\cdot\bar{r}\right)} \Psi(\bar{r}, t) \\ &\text{with } \bar{r} = \bar{r}' + \bar{v}\cdot t \end{aligned}}$$

Resolution of Landè's pseudo paradox

in dim = 1

$$\Psi'(x',t') = e^{i\frac{m}{\hbar}\left(\frac{v^2 t}{2} - vx\right)}\,\Psi(x,t)$$

for a plane wave $\Psi(x,t) = e^{2i\pi\left(\frac{x}{\lambda} - \nu t\right)}$

$$x = x' + vt \quad , \quad t = t'$$

$$\Psi'(x',t') = e^{i\frac{m}{\hbar}\left(\frac{v^2 t'}{2} - v(x'+vt')\right)}\, e^{2i\pi\left(\frac{x'+vt'}{\lambda} - \nu t'\right)} =$$

$$= e^{2i\pi\left(\frac{x'}{\lambda'} - \nu' t'\right)}$$

$$\boxed{\begin{aligned} \frac{1}{\lambda'} &= \frac{1}{\lambda} - \frac{mv}{h} \\ \nu' &= \nu - \frac{v}{\lambda} + \frac{mv^2}{2h} \end{aligned}}$$

Quantum Doppler effect for a Schrödinger wave

Limits

- $h \to 0$ - a point particle, not a wave
- $m \to 0$ - classical Doppler effect (but the Schrödinger equation does not exist!)

$\Rightarrow$ In a wave - particle duality the wave is not a classical wave, (real function), but a quantum wave which is a complex function.