III GALILEAN AND GAUGE TRANSFORMATIONS IN QUANTUM MECHANICS

§ 1. Landé's – Lévy-Leblond's pseudoparadox

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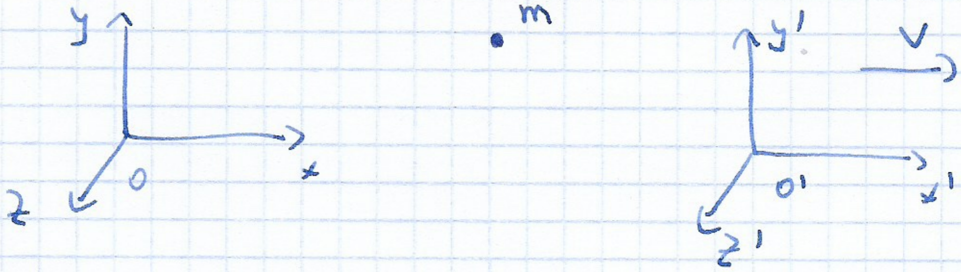


$$\boxed{\begin{aligned} x' &= x - vt \\ t' &= t \end{aligned}}$$

Galilean transformation

$$\boxed{p' = p - mv}$$

Let's check de Broglie and Einstein hypothesis

de Broglie

$$\boxed{p = \frac{h}{\lambda}}$$

Einstein

$$\boxed{E = h\nu}$$

Momentum transformation modifies de Broglie wave length

$$p' = \frac{h}{\lambda'}$$

$$p - mv = \frac{h}{\lambda'}$$

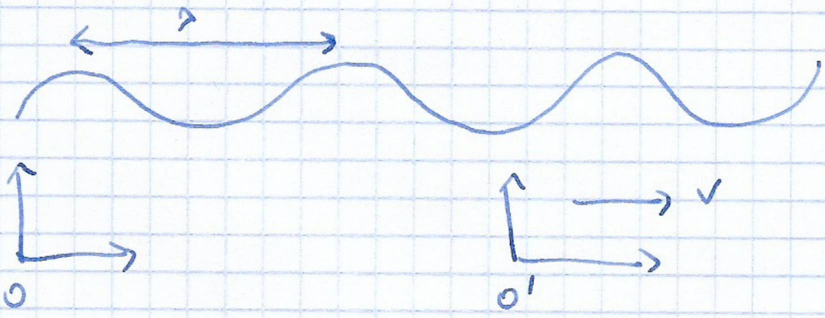
$$\frac{h}{\lambda} - mv = \frac{h}{\lambda'} \quad \longrightarrow \quad \boxed{\frac{1}{\lambda'} = \frac{1}{\lambda} - \frac{mv}{h}}$$

quantum Doppler effect

In classical wave physics $\lambda' = \lambda$!

Indeed, for a plane wave

$$\psi(x,t) = A \sin\left[2\pi\left(\frac{x}{\lambda} - \nu t\right)\right] \quad \text{in } O$$



in O' $\quad \psi'(x',t') = \psi(x,t)$

$$\psi'(x',t') = \psi(x,t) = A \sin\left[2\pi\left(\frac{x'+vt'}{\lambda} - \nu t'\right)\right] =$$

$$= A \sin\left[2\pi\left(\frac{x'}{\lambda} - \left(\nu - \frac{V}{\lambda}\right)t'\right)\right]$$

therefore

$$\boxed{\begin{aligned} \lambda' &= \lambda \\ \nu' &= \nu - \frac{V}{\lambda} \end{aligned}}$$

nonrelativistic Doppler effect

Einstein relation:

$$E = \frac{p^2}{2m} \qquad E' = \frac{p'^2}{2m} = \frac{(p - mV)^2}{2m} = \frac{p^2}{2m} - Vp + \frac{mV^2}{2}$$

Einstein plus Doppler effect

$$E' = h\nu' = h\nu - \frac{hV}{\lambda} = h\nu - pV$$

§2. Galilean transformation

m – mass of the particle

Wigner operator[*] $\hat{u}(\bar{v},t)$ is defined

$$\boxed{\begin{aligned} \hat{u}^+(\bar{v},t)\,\hat{\bar{r}}\,\hat{u}(\bar{v},t) &= \hat{\bar{r}} - \bar{v}t \\ \hat{u}^+(\bar{v},t)\,\hat{\bar{p}}\,\hat{u}(\bar{v},t) &= \hat{\bar{p}} - m\bar{v} \end{aligned}} \quad (*)$$

[*] Wigner theorem: for any transformation

$$G: \bar{R} \to \bar{R}' = G(\bar{R})$$

which conserves probability $|\langle\Psi|\Phi\rangle| = |\langle\Psi'|\Phi'\rangle|$ there is an operator $\hat{u} = \hat{u}_G$ such that

$$|\Psi'\rangle = \hat{u}\,|\Psi\rangle,$$

which is unitary ~~or~~ or anti unitary. $\square$

We assume that

$$\boxed{\hat{u}(\bar{v},t) = \hat{u}_p(\bar{v},t)\cdot\hat{u}_r(\bar{v},t)}$$

where $\hat{u}_p$ is a function of $\hat{\bar{p}}$ and $\hat{u}_r$ is a function of $\hat{\bar{r}}$.

If we assume

$$\boxed{\begin{aligned} \hat{u}_p^+(\bar{v},t)\,\hat{\bar{r}}\,\hat{u}_p(\bar{v},t) &= \hat{\bar{r}} - \bar{v}t \\ \hat{u}_r^+(\bar{v},t)\,\hat{\bar{p}}\,\hat{u}_r(\bar{v},t) &= \hat{\bar{p}} - m\bar{v} \end{aligned}}$$

equations (*) are satisfied.

•) $\hat{U}_p(\bar{v},t)$ is a translation operator of $\hat{\bar{r}}$ $\qquad \bar{a} = \bar{v}\cdot t$ ↑ translation

$$\boxed{\hat{U}_p(\bar{v},t) = e^{i\gamma_p(\bar{v},t) + i\frac{\bar{v}\cdot\hat{\bar{p}}}{\hbar}t}}$$

↑ arbitrary phase

•) $\hat{U}_r(\bar{v},t)$ – translates $\hat{\bar{p}}$ but leaves $\hat{\bar{r}}$ invariant $\qquad \bar{p}_0 = m\bar{v}$

$$\boxed{\hat{U}_r(\bar{v},t) = e^{i\gamma_r(\bar{v},t) - i\frac{m\bar{v}\cdot\hat{\bar{r}}}{\hbar}}}$$

using $e^{\hat{A}}e^{\hat{B}} = e^{\frac{1}{2}[\hat{A},\hat{B}]}e^{\hat{A}+\hat{B}}$ for $[\hat{A},\hat{B}] \in \mathbb{C}$

$$\hat{U}(\bar{v},t) = e^{i\gamma_p(\bar{v},t) + i\frac{\bar{v}\cdot\hat{\bar{p}}}{\hbar}t}\, e^{i\gamma_r(\bar{v},t) - i\frac{m\bar{v}\cdot\hat{\bar{r}}}{\hbar}}$$

$$\left[i\frac{\bar{v}\cdot\hat{\bar{p}}}{\hbar}t, -i\frac{m\bar{v}\cdot\hat{\bar{r}}}{\hbar}\right] = \frac{mt}{\hbar^2}\left[\bar{v}\cdot\hat{\bar{p}}, \bar{v}\cdot\hat{\bar{r}}\right] = \frac{mt}{\hbar^2}\sum_{i,j=1}^{3} v_i v_j \underbrace{[\hat{p}_i, \hat{r}_j]}_{-i\hbar\delta_{ij}} =$$

$$= \frac{mt}{\hbar^2}(-i\hbar)\sum_{i=1}^{3} v_i^2 = -i\frac{mv^2}{\hbar}t$$

$$\hat{U}(\bar{v},t) = e^{i\left(\gamma_p + \gamma_r - \frac{1}{2}\frac{mv^2}{\hbar}t\right)}\, e^{\frac{i}{\hbar}(\hat{\bar{p}}\cdot t - m\hat{\bar{r}})\cdot\bar{v}}$$

(*) We choose $\gamma \equiv \gamma_p + \gamma_r = \frac{1}{2}\frac{mv^2t}{\hbar}$ and get

$$\boxed{\hat{U}(\bar{v},t) = e^{\frac{i}{\hbar}(\hat{\bar{p}}t - m\hat{\bar{r}})\cdot\bar{v}}}$$

(*) this is only compatible choice with Galilean covariance

e.g. for free particle $\qquad \hat{H} = \frac{\hat{\bar{p}}^2}{2m}$

$$0 = [\hat{U}, \hat{H}] + i\partial_t\hat{U} \Rightarrow \frac{d\gamma(\bar{v},t)}{dt} = \frac{1}{2}mv^2$$

$$\rightarrow \quad \gamma(\bar{v},t) = \frac{1}{2}mv^2t + \underbrace{\gamma_0(\bar{v})}_{=0}$$

The wave function transformation

$$t_T = t$$

$$\psi_T(\bar r_T, t_T) = \langle \bar r_T | \hat u(\bar v, t) | \Psi(t)\rangle =$$

$$\hat u = \hat u_p \hat u_r$$

$$= \langle \bar r_T | \hat u_p \; \hat u_r | \Psi(t)\rangle =$$

$$= \int dr'' \int \frac{dp}{(2\pi\hbar)^{d/2}} e^{i\gamma} \langle \bar r_T | e^{i\frac{\bar v\cdot\hat{\bar p}}{\hbar}t} |\bar p\rangle\langle\bar p| e^{-i\frac{m\bar v\cdot\hat{\bar r}}{\hbar}} |\bar r''\rangle\underbrace{\langle \bar r''|\Psi(t)\rangle}_{\Psi(\bar r'', t)}$$

$$= \int dr''\int \frac{dp}{(2\pi\hbar)^{d/2}} e^{i\gamma}\, e^{i\frac{\bar v\cdot\bar p\, t}{\hbar}}\, e^{-i\frac{m\bar v\cdot \bar r''}{\hbar}} \underbrace{\langle \bar r_T|\bar p\rangle}_{\frac{e^{i\frac{\bar p\cdot\bar r_T}{\hbar}}}{(2\pi\hbar)^{d/2}}}\underbrace{\langle\bar p|\bar r''\rangle}_{\frac{e^{-i\frac{\bar p\cdot\bar r''}{\hbar}}}{(2\pi\hbar)^{d/2}}}\langle\bar r''|\Psi(t)\rangle$$

$$= \int dr''\int\frac{dp}{(2\pi\hbar)^d}\, e^{i\gamma - \frac{i}{\hbar}m\bar v\cdot\bar r''}\, e^{\frac{i}{\hbar}\bar p(\bar v\cdot t + \bar r_T - \bar r'')} =$$

$$= \int dr''\, e^{i\gamma - \frac{i}{\hbar} m\bar v\cdot\bar r''}\,\delta(\bar v\cdot t + \bar r_T - \bar r'') =$$

$$= e^{i\gamma - \frac{i}{\hbar} m\bar v(\bar v\cdot t + \bar r_T)} = \qquad \gamma = \frac{1}{2} m v^2 t$$

$$= e^{\frac{i}{\hbar}(\frac{1}{2} mv^2 t - mv^2 t - m\bar v\cdot\bar r_T)} =$$

$$= e^{\frac{i}{\hbar}(-\frac{1}{2} mv^2t - m\bar v\cdot(\bar r - \bar v t))} =$$

$$= e^{\frac{i}{\hbar}(\frac{1}{2} mv^2 t - m\bar v\cdot\bar r)} \qquad \bar r = \bar r_T + \bar v\cdot t$$

Galilean transformation

$$\hat u(\bar v,t) = e^{\frac{i}{\hbar}(\hat{\bar p}\, t - m\hat{\bar r})\cdot\bar v}$$

$$\hat{\bar r}_T = \hat u^+ \hat{\bar r}\,\hat u = \hat{\bar r} - \bar v\, t$$

$$\hat{\bar p}_T = \hat u^+\hat{\bar p}\,\hat u = \hat{\bar p} - m\bar v$$

$$\psi_T(\bar r_T, t) = e^{\frac{i}{\hbar}(\frac{mv^2}{2}t - m\bar v\cdot\bar r)}\,\Psi(\bar r, t)$$

with $\bar r = \bar r_T + \bar v\cdot t$

Note: $[\hat r_{Ti}, \hat p_{Tj}] = [r_i, p_j] = i\hbar\delta_{ij}$ – canonical

but in O and O' position and momentum operators are represented differently!

Resolution of Landè's pseudo paradox

in dim = 1

$$\psi'(x',t') = e^{i\frac{m}{\hbar}\left(\frac{v^2 t}{2} - vx\right)}\,\psi(x,t)$$

for a plane wave $\quad \psi(x,t) = e^{2i\pi\left(\frac{x}{\lambda} - \nu t\right)}$

$$x = x' + vt \quad , \quad t = t'$$

$$\psi'(x',t') = e^{i\frac{m}{\hbar}\left(\frac{v^2 t'}{2} - v(x'+vt')\right)}\, e^{2i\pi\left(\frac{x'+vt'}{\lambda} - \nu t'\right)} =$$

$$= e^{2i\pi\left(\frac{x'}{\lambda'} - \nu' t'\right)}$$

$$\boxed{\begin{aligned} \frac{1}{\lambda'} &= \frac{1}{\lambda} - \frac{mv}{h} \\ \nu' &= \nu - \frac{v}{\lambda} + \frac{mv^2}{2h} \end{aligned}}$$

<u>Quantum Doppler effect</u>

<u>for a Schrödinger wave</u>

<u>Limits</u>

- $h \to 0$ – a point particle, not a wave
- $m \to 0$ – classical Doppler effect (but the Schrödinger equation does not exist!)

$\Rightarrow$ In a wave – particle duality the wave is not a classical wave, real function, but a quantum wave which is a <u>complex function</u>.

§ 3. Non-equivalent representations of canonical commutation relations in quantum mechanics

$\boxed{d=1}$ (R. Shankar, QM, p. 208-212.)

In quantum mechanics independend classical observable position x and momentum p are represented by hermitian operators $\hat{x}$ and $\hat{p}$ which obeys

$$\boxed{[\hat{x}, \hat{p}] = i\hbar}$$

If we have a classical function $\omega(x,p)$, then in QM we have

$$\hat{\Omega} = \omega(x \to \hat{x}, p \to \hat{p}).$$

In a position representation $\{|x\rangle\}$

$$\hat{x} \longrightarrow x \qquad \hat{x}|x\rangle = x|x\rangle$$

$$\hat{p} \longrightarrow -i\hbar \frac{d}{dx}$$

check out

$$\forall f(x): \boxed{[\hat{x}, \hat{p}] f(x)} = \left(x\left(-i\hbar \frac{d}{dx}\right) - \left(-i\hbar \frac{d}{dx}\right) x \right) f(x) =$$

$$= \left(\cancel{-i\hbar x \frac{d}{dx}} + i\hbar + \cancel{i\hbar x \frac{d}{dx}} \right) f(x) = \boxed{i\hbar f(x)}$$

□

The equation $\hat{p}|p\rangle = p|p\rangle$ in a position representation

$$-i\hbar \frac{d}{dx} \psi_p(x) = p\, \psi_p(x)$$

solution $\boxed{\psi_p(x) = \frac{1}{\sqrt{2\pi\hbar}} e^{\frac{ipx}{\hbar}}}$

plane wave

However, there is a different representation in $\{|x\rangle\}$

$$\hat{x} \to x$$

$$\hat{p} \to -i\hbar \frac{d}{dx} + \chi(x)$$

Indeed,

$$\forall f(x): \boxed{[\hat{x}, \hat{p}] f(x)} = \left(x\left(-i\hbar \frac{d}{dx}\right) + x\chi(x) - \left(\left(-i\hbar\frac{d}{dx}\right) + \chi(x)\right)x \right) f(x) =$$

$$= \left(\cancel{-i\hbar x \frac{d}{dx}} + \cancel{x\chi(x)} + i\hbar + \cancel{i\hbar x \frac{d}{dx}} - \cancel{x\chi(x)}\right) f(x)$$

$$= \boxed{i\hbar f(x)}$$

$\square$

The equation $\hat{p}|p\rangle = p|p\rangle$ in the position representation reads

$$\left[-i\hbar \frac{d}{dx} + \chi(x)\right] \psi_p(x) = p\,\psi_p(x)$$

The solution is <u>not</u> a plane wave!

This change of <u>a operator representation</u> can be viewed as a unitary transformation of the base $\{|x\rangle\}$

$$\boxed{|x\rangle \to |\tilde{x}\rangle = e^{i\frac{f(\hat{x})}{\hbar}} |x\rangle = \underbrace{e^{\frac{i f(x)}{\hbar}}}_{\hat{u}(x)} |x\rangle \quad \text{where} \quad f(x) = \int^{x} \chi(x')\,dx'}$$

Indeed, in a new basis

$$\boxed{\langle \tilde{x} | \hat{x} | \tilde{x}' \rangle} = \langle x | e^{-i\frac{\rho(x)}{\hbar}} \hat{x} \, e^{i\frac{\rho(x)}{\hbar}} | x' \rangle =$$

$$= e^{-\frac{i}{\hbar}(\rho(x) - \rho(x'))} \underbrace{\langle x | \hat{x} | x' \rangle}_{x \langle x | x' \rangle = x\,\delta(x - x')} =$$

$$= \boxed{x\,\delta(x - x')}$$

$$\boxed{\langle \tilde{x} | \hat{p} | \tilde{x}' \rangle} = \langle x | e^{-i\frac{\rho(x)}{\hbar}} \frac{\hbar}{i}\frac{d}{dx'} e^{i\frac{\rho(x')}{\hbar}} | x' \rangle =$$

$$= e^{-i\frac{\rho(x)}{\hbar}} \langle x | \underbrace{\frac{d\rho(x')}{dx}}_{\chi(x)} + \frac{\hbar}{i}\frac{d}{dx'} | x' \rangle \, e^{i\frac{\rho(x')}{\hbar}} =$$

$$= e^{-\frac{i}{\hbar}(\rho(x) - \rho(x'))} \langle x | -i\hbar\frac{d}{dx} + \chi(x) | x' \rangle =$$

$$= e^{-\frac{i}{\hbar}(\rho(x) - \rho(x'))} \delta(x - x') \left(-i\hbar\frac{d}{dx} + \chi(x) \right) =$$

$$= \boxed{\delta(x - x') \left(-i\hbar\frac{d}{dx} + \chi(x) \right)}$$

Conclusions: The basis $\{|x\rangle\}$ is not unique. Multiplication by $e^{i\rho(x)/\hbar}$ conserves normalization and orthogonality. Matrix elements of $\hat{p}$ are different for different functions $\chi(x)$. Standard choice is $\chi(x) = 0$.

The change of $\chi(x)$ does not change the physical content. The novelty is that we change the basis $|x\rangle$ and not the operator representation $x \to$ [illegible] or $p \to$ [illegible] etc. Two operators $\hat{\omega}(x, -i\hbar\frac{d}{dx})$ and $\hat{\omega}(x, -i\hbar\frac{d}{dx} + \chi(x))$ have the same spectrum corresponding to $\hat{\Omega}(\hat{x}, \hat{p})$.

$\boxed{d = 2 \text{ or } 3}$ In higher dimensions $\mathbb{R}^2$ or $\mathbb{R}^3$

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$$|\vec r\rangle = |x_1, x_2, x_3\rangle \,, \quad |\vec p\rangle = |p_1, p_2, p_3\rangle$$

$$\hat{\vec r}\,|\vec r\rangle = \vec r\,|\vec r\rangle \qquad \hat{\vec p}\,|\vec p\rangle = \vec p\,|\vec p\rangle$$

$$\begin{pmatrix}\hat x_1\\ \hat x_2\\ \hat x_3\end{pmatrix}|x_1, x_2, x_3\rangle = \begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}|x_1, x_2, x_3\rangle \qquad \langle \vec r|\vec p\rangle = \frac{e^{i\,\vec p\cdot\vec r/\hbar}}{(\sqrt{2\pi\hbar})^3}$$

non-equivalent representations

$i = 1, 2, 3 \ldots d$

$$\hat X_i \longrightarrow x_i$$

$$\hat P_i \longrightarrow -i\hbar\frac{\partial}{\partial x_i} + \chi_i(x_1, x_2, x_3)$$

$$|\vec r\rangle \longrightarrow |\tilde{\vec r}\rangle = \hat u(\vec r)\,|\vec r\rangle = e^{i\frac{\rho(\vec r)}{\hbar}}\,|\vec r\rangle$$

where $\rho(\vec r) = \int^{\vec r} \vec\chi(\vec r')\cdot d\vec r' = \sum_{i=1}^{d}\int^{\vec r}\chi_i(\vec r)\,dx_i$

$$\tilde{\hat X}_i = \hat u\,\hat x_i\,\hat u^{-1}$$
$$\tilde{\hat P}_i = \hat u\,\hat P_i\,\hat u^{-1}$$

Indeed,

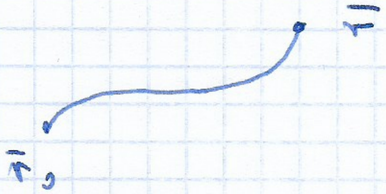
$$[\tilde X_i, \tilde P_j] = x_i\left(-i\hbar\frac{\partial}{\partial x_j}\right) + x_i\chi_j - \left(-i\hbar\frac{\partial}{\partial x_j}\right)x_i - \chi_j x_i \boxed{= \delta_{ij}\, i\hbar}$$

$\square$

But

$$\rho(\vec r) = \int_{\vec r_0}^{\vec r} \vec\chi(\vec r')\cdot d\vec r'$$

can not depend on the path!



In other words, $d f(\bar r)$ must be an <u>exact</u> differentiable (form)

In $d=1$ it is always possible up to an additive constant

$$d f = \chi(x)\,dx \quad \rightarrow \quad f(x) = \int^x \chi(x')\,dx' + \text{cost.}$$

In $d>1$ if $f(\bar r)$ exists then

$$d f(\bar r) = \bar\nabla f(\bar r)\cdot d\bar r = \sum_{i=1}^{d} \underbrace{\frac{\partial f(\bar r)}{\partial x_i}}_{\chi_i(\bar r)} dx_i = \sum_{i=1}^{d} \chi_i(\bar r)\,dx_i$$

by computing higher derivatives

$$\frac{\partial \chi_i}{\partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i} = \frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial \chi_j}{\partial x_i}$$

or $\boxed{\frac{\partial \chi_i}{\partial x_j} - \frac{\partial \chi_j}{\partial x_i} = 0} \iff \boxed{\bar\nabla \times \bar\chi = 0}$

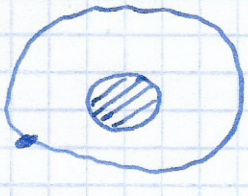
<u>von Neumann theorem</u>

<u>This condition makes the different representations equivalent, i.e. there is a unitary transformation</u>

<u>Remarks</u>

1) A useful trick to solve $\hat H = \frac{1}{2m}\left[\hat{\bar p} - \bar\chi(\bar r)\right]^2$ with $\bar\nabla\times\bar\chi = 0$ is to introduce $\tilde\psi(\bar r) = e^{\frac{i}{\hbar}\int^{\bar r} \bar\chi(\bar r')\cdot d\bar r'}\,\psi(\bar r)$, which amounts to chose an equivalent representation with convinient commutation relations.

2) In a <u>not simple connected spaces</u> these representations <u>are not equivalent</u>



a simple closed curve can not be continuously shrunk to a point

A representation $\{\hat{\tilde{r}}_i, \hat{\tilde{p}}_i\}$ exists but is not connected to the existing representation $\{\hat{\bar{r}}, \hat{\bar{p}}_i\}$ by a unitary operator.

In not-simple connected spaces the condition

$$\bar{\nabla} \times \bar{\chi} = 0$$

implies locally that $\bar{\chi}(\bar{r}) = \bar{\nabla} \rho(\bar{r})$, $d\rho(\bar{r})$ is closed $(d^2\rho(\bar{r}) = 0)$ but it is no longer an exact differential form, i.e. no differentiable function $\rho(\bar{r})$ exists such that $\bar{\chi}(\bar{r}) = \bar{\nabla}\rho(\bar{r})$ in <u>all</u> the configuration space. As a consequence $\int_{\bar{r}_0}^{\bar{r}} \bar{\chi}(\bar{r}) \cdot d\bar{r}$ is <u>not single valued</u> and

$$\hat{u} = e^{\frac{i}{\hbar}\int_{\bar{r}_0}^{\bar{r}} \bar{\chi}(\bar{r}) \cdot d\bar{r}}$$ <u>does not exist</u>,

This mathematical circumstances lead to very interesting, topological effects in physical quantum-mechanical systems.