$\boxed{d = 2 \text{ or } 3}$ In higher dimensions $\mathbb{R}^2$ or $\mathbb{R}^3$

$\boxed{\text{AJP 75, 268 (2007)}}$

$$|\bar{r}\rangle = |x_1, x_2, x_3\rangle \, , \quad |\bar{p}\rangle = |p_1, p_2, p_3\rangle$$

$$\hat{\bar{r}}\,|\bar{r}\rangle = \bar{r}\,|\bar{r}\rangle \qquad \hat{\bar{p}}\,|\bar{p}\rangle = \bar{p}\,|\bar{p}\rangle$$

$$\begin{pmatrix}\hat{x}_1\\ \hat{x}_2\\ \hat{x}_3\end{pmatrix}|x_1, x_2, x_3\rangle = \begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}|x_1, x_2, x_3\rangle \qquad \langle \bar{r}|\bar{p}\rangle = \frac{e^{i\,\bar{p}\cdot\bar{r}/\hbar}}{\left(\sqrt{2\pi\hbar}\right)^3}$$

non-equivalent representations

$i = 1, 2, 3 \ldots d$

$$\boxed{\begin{aligned} &\hat{X}_i \longrightarrow x_i \\ &\hat{P}_i \longrightarrow -i\hbar\frac{\partial}{\partial x_i} + \chi_i(x_1, x_2, x_3) \\ &|\bar{r}\rangle \longrightarrow |\tilde{\bar{r}}\rangle = \hat{u}(\bar{r})|\bar{r}\rangle = e^{i\frac{\rho(\bar{r})}{\hbar}}|\bar{r}\rangle \\ &\text{where} \quad \rho(\bar{r}) = \int^{\bar{r}} \bar{\chi}(\bar{r}')\cdot d\bar{r}' = \sum_{i=1}^{3}\int^{\bar{r}} \chi_i(\bar{r}')\,dx_i' \end{aligned}}$$

$$\boxed{\begin{aligned} \hat{\tilde{X}}_i &= \hat{u}\,\hat{x}_i\,\hat{u}^{-1} \\ \hat{\tilde{P}}_i &= \hat{u}\,\hat{p}_i\,\hat{u}^{-1} \end{aligned}}$$

Indeed,

$$[\tilde{X}_i, \tilde{P}_j] = x_i\left(-i\hbar\frac{\partial}{\partial x_j}\right) + x_i\chi_j -$$

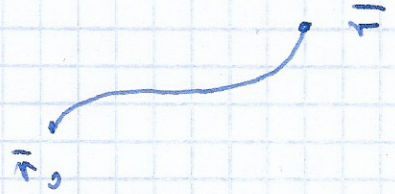
$$-\left(-i\hbar\frac{\partial}{\partial x_j}\right)x_i - \chi_j x_i \; \boxed{= \delta_{ij}\, i\hbar}$$

$\square$

But

$$\rho(\bar{r}) = \int_{\bar{r}_0}^{\bar{r}} \bar{\chi}(\bar{r}')\cdot d\bar{r}'$$

can not depend on the path!



In other words, $d\rho(\bar{r})$ must be an exact differentiable (form)

In $d=1$ it is always possible up to an additive constant

$$d\rho = \chi(x)\,dx \quad \rightarrow \quad \rho(x) = \int^x \chi(x')\,dx' + \text{cost.}$$

In $d>1$ if $\rho(\bar{r})$ exists then

$$d\rho(\bar{r}) = \bar{\nabla}\rho(\bar{r})\cdot d\bar{r} = \sum_{i=1}^{d} \underbrace{\frac{\partial \rho(\bar{r})}{\partial x_i}}_{\chi_i(\bar{r})} dx_i = \sum_{i=1}^{d} \chi_i(\bar{r})\,dx_i$$

by computing higher derivatives

$$\frac{\partial \chi_i}{\partial x_j} = \frac{\partial^2 \rho}{\partial x_j \partial x_i} = \frac{\partial^2 \rho}{\partial x_i \partial x_j} = \frac{\partial \chi_j}{\partial x_i}$$

or $\boxed{\dfrac{\partial \chi_i}{\partial x_j} - \dfrac{\partial \chi_j}{\partial x_i} = 0} \iff \boxed{\bar{\nabla}\times\bar{\chi} = 0}$

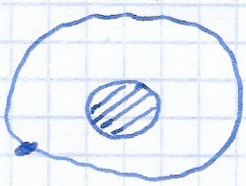
<u>von Neumann theorem</u>

<u>This condition makes the different representations equivalent, i.e. there is a unitary transformation</u>

<u>Remarks</u>

1) A useful trick to solve $\hat{H} = \frac{1}{2m}\left[\hat{\bar{p}} - \bar{\chi}(\bar{r})\right]^2$ with $\bar{\nabla}\times\bar{\chi}=0$ is to introduce $\tilde{\psi}(\bar{r}) = e^{\frac{i}{\hbar}\int^{\bar{r}} \bar{\chi}(\bar{r}')\cdot d\bar{r}'}\psi(\bar{r})$, which amounts to chose an equivalent representation with convinient commutation relations.

2) In a <u>not simple connected spaces</u> these representations <u>are not equivalent</u>



a simple closed curve can not be continuously shrunk to a point

A representation $\{\hat{\bar{r}}_i, \hat{\bar{p}}_i\}$ exists but is not connected to the existing representation $\{\hat{\bar{r}}, \hat{\bar{p}}_i\}$ by a unitary operator.

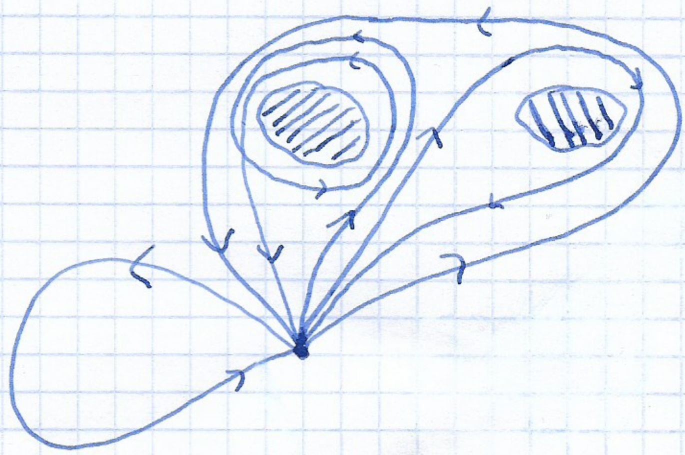
In not-simple connected spaces the condition

$$\bar{\nabla} \times \bar{\chi} = 0$$

implies locally that $\bar{\chi}(\bar{r}) = \bar{\nabla} \rho(\bar{r})$, $d\rho(\bar{r})$ is closed $(d^2 \rho(\bar{r}) = 0)$ but it is no longer an exact differential form, i.e. no differentiable function $\rho(\bar{r})$ exists such that $\bar{\chi}(\bar{r}) = \bar{\nabla} \rho(\bar{r})$ in <u>all</u> the configuration space. As a consequence $\int_{\bar{r}_0}^{\bar{r}} \bar{\chi}(\bar{r}) \cdot d\bar{r}$ is <u>not simple valued</u> and

$\hat{u} = e^{\frac{i}{\hbar} \int_{\bar{r}_0}^{\bar{r}} \bar{\chi}(\bar{r}) \cdot d\bar{r}}$ <u>does not exist</u>,

This mathematical circumstances lead to very interesting, topological effects in physical quantum-mechanical systems.



§4. Electromagnetic gauge representations in quantum mechanics

•) Scalar and vector potentials in classical electrodynamics

Two Maxwell equations

$$\begin{cases} \bar{\nabla} \cdot \bar{B} = 0 \\ \bar{\nabla} \times \bar{E} + \dfrac{\partial \bar{B}}{\partial t} = 0 \end{cases}$$

can be satisfied (as identities) by settings

$$\boxed{\begin{aligned} \bar{E} &= -\bar{\nabla}\phi - \frac{\partial \bar{A}}{\partial t} \\ \bar{B} &= \bar{\nabla} \times \phi \end{aligned}}$$

$\bar{A}$ – vector potential

ϕ – scalar potential

vector function

Indeed, $\bar{\nabla} \cdot (\bar{\nabla} \times \bar{A}) = 0$ for any $\bar{A}$!

$$\bar{\nabla} \times \bar{E} + \frac{\partial}{\partial t} \bar{\nabla} \times \bar{A} = 0$$

sign convention

$$\bar{\nabla} \times \left(\bar{E} + \frac{\partial \bar{A}}{\partial t} \right) = 0 \rightarrow \bar{E} = -\frac{\partial \bar{A}}{\partial t} - \bar{\nabla}\phi$$

because $\bar{\nabla} \times (\bar{\nabla}\phi) = 0$

scalar function

Taking other two Maxwell equations

$$\begin{cases} \bar{\nabla} \cdot \bar{E} = \dfrac{\rho}{\varepsilon_0} \\ \bar{\nabla} \times \bar{B} = \mu_0 \vec{j} + \mu_0 \varepsilon_0 \dfrac{\partial \bar{E}}{\partial t} \end{cases}$$

we can find equation on ϕ and $\bar{A}$ by ρ and $\vec{j}$

Indeed,

$$\bar\nabla\cdot\bar E = -\frac{\partial}{\partial t}\bar\nabla\cdot\bar A - \bar\nabla\cdot\bar\nabla\phi = \rho/\varepsilon_0$$

$$\boxed{\bar\nabla^2\phi + \frac{\partial}{\partial t}(\bar\nabla\cdot\bar A) = -\rho/\varepsilon_0}$$

$$\bar\nabla\times\bar B - \mu_0\varepsilon_0\frac{\partial\bar E}{\partial t} = \bar\nabla\cdot(\bar\nabla\cdot\bar A) - \bar\nabla^2\bar A + $$
$$+\mu_0\varepsilon_0\left(\frac{\partial^2\bar A}{\partial t^2} + \frac{\partial}{\partial t}\bar\nabla\phi\right) = \mu_0\vec j$$

$$\boxed{\bar\nabla^2\bar A - \mu_0\varepsilon_0\frac{\partial^2\bar A}{\partial t^2} - \bar\nabla\left(\bar\nabla\cdot\bar A + \varepsilon_0\mu_0\frac{\partial\phi}{\partial t}\right) = -\mu_0\vec j}$$

•) <u>gauge freedom in classical electrodynamics</u>

$$(\bar E,\bar B)\leftrightarrow \boxed{\begin{aligned}\bar A &\to \bar A' = \bar A + \bar\nabla\chi\\ \phi &\to \phi' = \phi - \frac{\partial\chi}{\partial t}\end{aligned}} \leftrightarrow (\bar E,\bar B)$$

$\chi = \chi(\bar r, t)$ – any scalar function

Indeed, $\bar B' = \bar\nabla\times\bar A' = \bar\nabla\times\bar A + \underbrace{\bar\nabla\times\bar\nabla\chi}_{0} = \bar B$

$$\bar E' = -\frac{\partial\bar A'}{\partial t} - \bar\nabla\phi' = -\frac{\partial\bar A}{\partial t} - \cancel{\bar\nabla\frac{\partial\chi}{\partial t}} - \bar\nabla\phi + \cancel{\bar\nabla\frac{\partial\chi}{\partial t}} =$$
$$= -\frac{\partial\bar A}{\partial t} - \bar\nabla\phi = \bar E$$

□

Since ϕ and $\bar A$ are determined up to a function (gauge) χ, they can not represent physical fields – can not be observed.

→ typical choice <u>Coulomb gauge</u>

$\underline{\bar\nabla\cdot\bar A = 0,\ \phi = 0}$ with $\bar A|_{\Sigma_0} = 0$

Then $\bar E$ and $\bar B$ determine $\bar A$ uniquely

→ uniform magnetic field

$$\bar{B} = (0, 0, B)$$

$$\bar{\nabla} \times \bar{A} = \begin{vmatrix} e_x & e_y & e_z \\ \partial_x & \partial_y & \partial_z \\ A_x & A_y & A_z \end{vmatrix}$$

$$\left.\begin{aligned} \bar{A} &= (0, Bx, 0) \\ \bar{A} &= (-By, 0, 0) \end{aligned}\right\} \text{ Landau gauge}$$

$$\bar{A} = \left(-\frac{By}{2}, \frac{Bx}{2}, 0\right) \quad \text{– symmetric gauge}$$

→ uniform electric field

$$\bar{E} = (0, 0, E)$$

$$\phi = -Ez, \quad \bar{A} = 0$$

$$\phi = 0, \quad \bar{A} = (0, 0, -Et)$$

•) Electromagnetic fields in quantum mechanics

In classical mechanics to get the correct form of the Lorenz force one introduce the vector (and scalar) potential to the Lagrangian

$$L(\bar{r}, \bar{v}) = \frac{1}{2} m v^2 + q \bar{v} \cdot \bar{A} - e\phi$$

or to the Hamiltonian

$$H(\bar{r}, \bar{p}) = \frac{(\bar{p} - e\bar{A})^2}{2m} + e\phi$$

In quantum mechanics we replace variables by operators

$$\bar{r} \to \hat{\bar{r}}$$

$$\bar{p} \to \hat{\bar{p}}$$

$$H \to \hat{H}$$

Comparing Hamiltonians

$$\boxed{\begin{aligned} \hat{H} &= \frac{\hat{\bar{p}}^2}{2m} \\ \hat{H} &= \frac{(\hat{\bar{p}} - q\bar{A})^2}{2m} \end{aligned}}$$

we can interpret that two different representation of momentum are used in a position representation

$\boxed{\hat{\bar{p}}}$ and $\boxed{\hat{\bar{p}} - q\bar{A} = \hat{\bar{\pi}}}$

mechanical kinematical momentum

generalized dynamical momentum

(classically, even when $\bar{v} = 0$ ($\bar{p} = 0$), $\bar{\pi} \neq 0$)

the corresponding change of the wave function

$$\boxed{\Psi(r,t) \longrightarrow e^{\frac{i}{\hbar} q \int^{\bar{r}} A(\bar{r}',t) \cdot d\bar{r}'} \Psi(\bar{r},t)}$$

and in the ~~state~~ vector $|\bar{r}\rangle$

$$\boxed{|\bar{r}\rangle \rightarrow e^{\frac{i}{\hbar} q \int^{\bar{r}} \bar{A}(\bar{r}',t) \cdot d\bar{r}'} |\bar{r}\rangle}$$

It is known as a minimal coupling procedure.

•) Electromagnetic gauge transformation in quantum mechanics

$$i\hbar \frac{\partial \Psi}{\partial t} = \frac{1}{2m}\left(\hat{p} - q\bar{A}\right)^2 \Psi + q\phi\Psi$$

$$i\hbar \frac{\partial \Psi'}{\partial t} = \frac{1}{2m}\left(\hat{p} - q\bar{A}'\right)^2 \Psi' + q\phi'\Psi'$$

$$\boxed{\begin{aligned} \bar{A}' &= \bar{A} + \bar{\nabla}\chi \\ \phi' &= \phi - \frac{\partial \chi}{\partial t} \end{aligned}}$$

$$\Psi \overset{?}{\longleftrightarrow} \Psi'$$

We show that $\boxed{\Psi'(\bar{r},t) = e^{+i\frac{q}{\hbar}\chi(\bar{r},t)}\,\Psi(\bar{r},t)}$

Proof

$$i\hbar \frac{\partial}{\partial t}\Psi' = i\hbar\left[i\frac{q}{\hbar}\frac{\partial \chi}{\partial t}\Psi + \frac{\partial \Psi}{\partial t}\right]e^{i\frac{q}{\hbar}\chi}$$

$$(\hat{p} - q\bar{A}')\Psi' = (-i\hbar\bar{\nabla} - q\bar{A} - q\bar{\nabla}\chi)e^{i\frac{q}{\hbar}\chi}\Psi =$$

$$= \left(-i\hbar\left(i\frac{q}{\hbar}\bar{\nabla}\chi\right)\Psi - i\hbar\bar{\nabla}\Psi - q\bar{A}\Psi - q\bar{\nabla}\chi\Psi\right)e^{i\frac{q}{\hbar}\chi} =$$

$$= \left(\cancel{q\bar{\nabla}\chi\Psi} - i\hbar\bar{\nabla}\Psi - q\bar{A}\Psi - \cancel{q\bar{\nabla}\chi\Psi}\right)e^{i\frac{q}{\hbar}\chi} = \left((\hat{p} - q\bar{A})\Psi\right)e^{i\frac{q}{\hbar}\chi}$$

$$(\bar{p} - q\bar{A}')^2\Psi' = \left((\bar{p} - q\bar{A})^2\Psi\right)e^{i\frac{q}{\hbar}\chi}$$

$$\boxed{i\hbar \frac{\partial \Psi'}{\partial t} = \frac{1}{2m}(\bar{p} - q\bar{A}')^2\Psi'}$$

$$\hookrightarrow\; i\hbar\left(\cancel{i\frac{q}{\hbar}\frac{\partial \chi}{\partial t}\Psi} + \frac{\partial \Psi}{\partial t}\right)e^{i\frac{q}{\hbar}\chi} = \frac{1}{2m}\left((\bar{p} - q\bar{A})^2\Psi\right)e^{i\frac{q}{\hbar}\chi} +$$

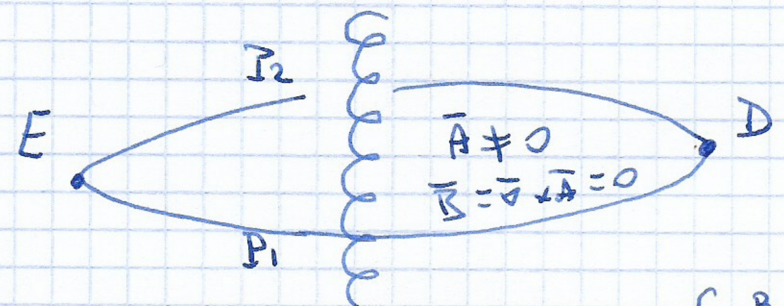
$$+ q\phi\Psi e^{i\frac{q}{\hbar}\chi} - \cancel{q\frac{\partial \chi}{\partial t}\Psi e^{i\frac{q}{\hbar}\chi}}$$

$$\boxed{i\hbar \frac{\partial \Psi}{\partial t} = \frac{1}{2m}(\bar{p} - q\bar{A})^2\Psi + q\phi\Psi}$$

□

§ 5. Aharonov - Bohm effect

Consider a motion of a charge around the infinite solenoid from point E to D



$$\bar{B} = \begin{cases} B\, e_z & \text{inside solenoid} \\ 0 & \text{outside} \end{cases} \quad \Leftrightarrow \quad \bar{A} = \begin{cases} \frac{B}{2} r\, \hat{e}_\varphi & \text{inside } r < R \\ \frac{B}{2}\frac{R^2}{r}\hat{e}_\varphi & \text{outside } r > R \end{cases}$$

Let detector is at point $\bar{r}$ and emitter at point $\bar{r}_0$

without $\bar{A}$ $\qquad \psi(\bar{r}) = \psi_{P_1}(r) + \psi_{P_2}(r)$

with $\bar{A}$ $\qquad$ ~~$\psi(\bar{r}) = \psi_{P_1}(1 + e^{i\frac{q}{\hbar}\int_{P_1}\bar{A}\cdot d\bar{r}}$~~

$$\psi(r) = \psi_{P_1}(\bar{r})\, e^{i\frac{q}{\hbar}\int_{P_1}\bar{A}\cdot d\bar{r}} + \psi_{P_2}(\bar{r})\, e^{i\frac{q}{\hbar}\int_{P_2}\bar{A}\cdot d\bar{r}}$$

the relative phases are modified and depend on particular paths.

$$\psi(\bar{r}) = e^{i\frac{q}{\hbar}\int_{P_2}\bar{A}\cdot d\bar{r}}\left[\psi_{P_1}(\bar{r})\, e^{i\frac{q}{\hbar}\oint_{C=P_1-P_2}\bar{A}\cdot d\bar{r}} + \psi_{P_2}(\bar{r})\right]$$

and

$$|\psi(\bar{r})|^2 = \left| e^{i\Delta\varphi_{12}}\, \psi_{P_1}(\bar{r}) + \psi_{P_2}(\bar{r})\right|^2$$

$$\Delta\varphi_{12} = \frac{q}{\hbar}\oint_C \bar{A}\cdot d\bar{r} = \frac{q}{\hbar}\int \bar{B}\cdot d\bar{S} = \frac{q}{\hbar}\Phi$$

magnetic flux

In quantum mechanics the vector potential affects the wave function and gives rise to shift of interference fringes

$$|\psi(\bar{r})|^2 = |\psi_{p_1}|^2 + |\psi_{p_2}|^2 + \psi_{p_1}^* \psi_{p_2} e^{-i\Delta\varphi_{12}} + \psi_{p_2}^* \psi_{p_1} e^{i\Delta\varphi_{12}}$$

$$\overset{\psi_p \in \mathbb{R}}{=} \psi_{p_1}^2 + \psi_{p_2}^2 + \underbrace{2\psi_{p_1}\psi_{p_2} \cos\left(\frac{e}{\hbar}\Phi\right)}$$

Remember, a point charge moves in a space where $\bar{B} = 0$!

~~A~~ infinite solenoid makes the space not-simple connected.

For future we note that if

$$e^{i\frac{e}{\hbar}\oint_C \bar{A}\cdot d\bar{r}} = e^{i\frac{e}{\hbar}\Phi} = e^{2\pi i m} \qquad m \in \mathbb{Z}$$

i.e. $$\boxed{\Phi = \frac{2\pi\hbar}{e} m = m\,\Phi_0}$$

$$\boxed{\Phi_0 = \frac{h}{e}} \quad \text{Df.}$$

the flux quantum

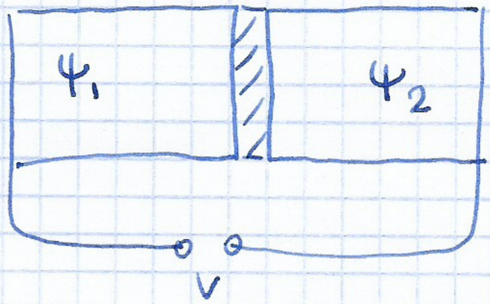
the solenoid is invisible

§6. Josephson junction

In a superconductor all electrons are described by a single wave function!

•) Consider two superconductors separated by an insulator in a potential difference V

Josephson junction



K – tunneling amplitude / coupling between these superconductors

$$i\hbar \frac{\partial \psi_1}{\partial t} = \frac{qV}{2}\psi_1 + K\psi_2$$

$$i\hbar \frac{\partial \psi_2}{\partial t} = -\frac{qV}{2}\psi_2 + K\psi_1$$

Assume the form of the wave functions

$$\psi_1 = \sqrt{\rho_1}\, e^{i\theta_1}$$

$$\psi_2 = \sqrt{\rho_2}\, e^{i\theta_2}$$

$$\delta \equiv \theta_2 - \theta_1$$

Then

$$\dot{\rho}_1 = \frac{2}{\hbar} K \sqrt{\rho_1 \rho_2}\, \sin\delta \quad = J \text{ – current from } 1 \to 2$$

$$\dot{\rho}_2 = -\frac{2}{\hbar} K \sqrt{\rho_1 \rho_2}\, \sin\delta \quad = -J \text{ – current from } 2 \to 1$$

$$\dot{\theta}_1 = \frac{K}{\hbar}\sqrt{\frac{\rho_2}{\rho_1}}\cos\delta - \frac{qV}{2\hbar}$$

$$\dot{\theta}_2 = \frac{K}{\hbar}\sqrt{\frac{\rho_1}{\rho_2}}\cos\delta + \frac{qV}{2\hbar}$$

$\Rightarrow$ $\boxed{J = J_0 \sin\delta}$

$$J_0 = \frac{2K\rho_0}{\hbar}$$

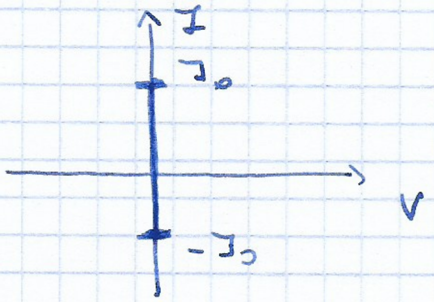
if $\rho_1 = \rho_2 = \rho_0 = \text{const.}$ due to a battery

$$\dot{\delta} = \dot{\theta}_2 - \dot{\theta}_1 = \frac{qV}{\hbar}$$

$$\delta(t) = \delta_0 + \frac{q}{\hbar}\int V(t)\,dt$$

DC for a constant $V = V_0 \rightarrow \delta(t) = \delta_0 + \frac{q}{\hbar} V_0 t$

↑ small $\hbar$



Anderson, Rowell 1963

at $V_0 = 0$ there is a Josephson current $\underline{J = J_0 \sin\delta_0}$ — thermodynamic current

if $V_0 \neq 0$ the current averages to zero

AC for a $V(t) = V_0 + v\cos\omega t$

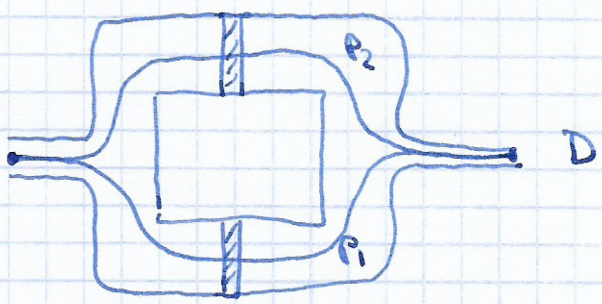
$$\delta(t) = \delta_0 + \frac{q}{\hbar} V_0 t + \frac{q}{\hbar}\frac{v}{\omega}\sin(\omega t)$$

$\sin(x + \Delta x) \simeq \sin x + \Delta x \cos x$

$$J = J_0\Big[\underbrace{\sin\big(\delta_0 + \tfrac{q}{\hbar} V_0 t\big)}_{\text{averages to zero}} + \underbrace{\tfrac{q}{\hbar}\tfrac{v}{\omega}\sin(\omega t)\cos\big(\delta_0 + \tfrac{q}{\hbar} V_0 t\big)}_{\text{contributes if } \omega = \frac{q}{\hbar} V_0}\Big]$$

Shapiro, 1963

•) consider two Josephson junction



the phase difference

$$\boxed{\delta_2 - \delta_1 = \frac{2q_e}{\hbar}\oint_C \bar{A}\cdot d\bar{r}}$$

Cooper pair $|k\uparrow, -k\downarrow\rangle$ of charge $Q = 2q_e$

$$\delta_2 - \delta_1 = \frac{2q_e}{\hbar}\Phi$$

↑ magnetic flux

Now $\delta_1 = \delta_0 + \frac{2e}{\hbar} \Phi$

$\delta_2 = \delta_0 - \frac{2e}{\hbar} \Phi$

$$J_{tot} = J_0 \left(\sin\left(\delta_0 + \frac{2e}{\hbar}\Phi\right) + \sin\left(\delta_0 - \frac{2e}{\hbar}\Phi\right)\right) =$$

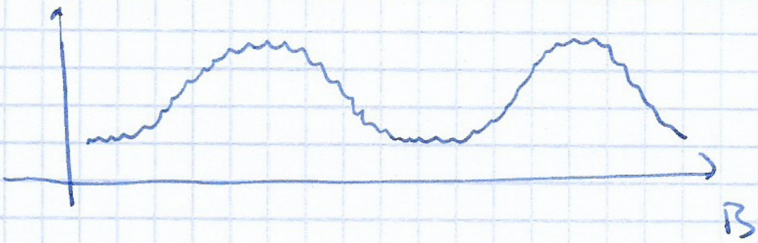
$$= J_0 \sin(\delta_0) \cos\left(\frac{2e\Phi}{\hbar}\right)$$

The maximal current is

$$J_{max} = J_0 \left| \cos\left(\frac{2e\Phi}{\hbar}\right)\right|$$

and oscillates with $\Phi = \int \vec{B} \cdot d\vec{S}$.

The maxima are at $\Phi = n \frac{\pi \hbar}{2e}$, $n \in \mathbb{Z}$



very sensitive device to detect a weak magnetic field → computer tomograph

medicine