Last lectures

$$\hat{\bar r} \longrightarrow \hat{\bar r}$$

$$\hat{\bar p} \longrightarrow \hat{\bar p} + \bar{\chi}(\bar r)$$

$$|\bar r\rangle \longrightarrow |\tilde{\bar r}\rangle = e^{i\frac{1}{\hbar}\int^{\bar r} \bar{\chi}(\bar r)\cdot d\bar r} |\bar r\rangle$$

if EM

$$\bar p \rightarrow \bar p - q\bar A$$

$$\psi(\bar r) \longrightarrow e^{i\frac{q}{\hbar}\int^{\bar r}\bar A\cdot d\bar r}\psi(\bar r)$$

$$\bar\nabla \times \bar\chi = 0$$

---

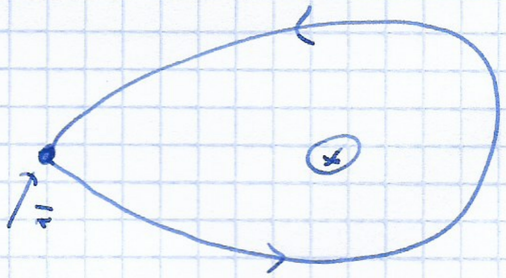
gauge transformation

$$\bar E = -\bar\nabla\phi - \frac{\partial \bar A}{\partial t} \qquad \bar A' = \bar A + \bar\nabla\chi$$

$$\bar B = \bar\nabla\times\bar A \qquad \phi' = \phi - \frac{\partial\chi}{\partial t}$$

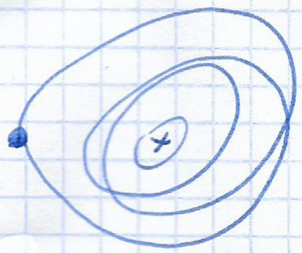
QM local gauge transformation $\qquad \psi'(\bar r,t) = e^{i\frac{q}{\hbar}\chi(\bar r,t)}\psi(\bar r,t)$

Aharonov - Bohm effect



$$\psi(\bar r) \longrightarrow e^{\frac{i}{\hbar}q\oint\bar A\cdot d\bar r}\psi(\bar r) = e^{\frac{i}{\hbar}q\Phi}\psi(\bar r)$$

topological quantity $\qquad \Phi = \int\bar B\cdot d\bar S = \oint\bar A\cdot d\bar r \qquad$ magnetic flux



$$\psi \rightarrow e^{\frac{i}{\hbar}q\,n\Phi}\psi$$

$n \in \mathbb{Z}$, - winding number

$$\boxed{\Phi_0 = \frac{h}{q}}$$

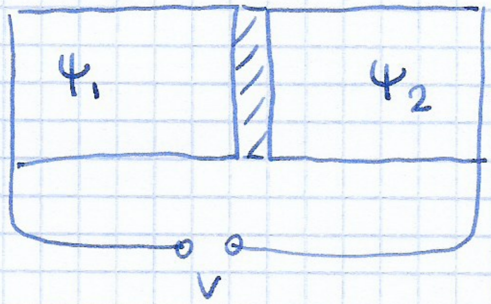
flux quantum

§ 6. Josephson junction

In a superconductor all electrons are described by a single wave function!

- Consider two superconductors separated by an insulator in a potential difference V

Josephson junction



K – tunnelin amplitude / coupling between these superconductors

$$i\hbar \frac{\partial \psi_1}{\partial t} = \frac{qV}{2}\psi_1 + K\psi_2$$

$$i\hbar \frac{\partial \psi_2}{\partial t} = -\frac{qV}{2}\psi_2 + K\psi_1$$

Assume the form of the wave functions

$$\psi_1 = \sqrt{\rho_1}\, e^{i\theta_1} \qquad \delta \equiv \theta_2 - \theta_1$$

$$\psi_2 = \sqrt{\rho_2}\, e^{i\theta_2}$$

Then

$$\dot{\rho}_1 = \frac{2}{\hbar} K \sqrt{\rho_1 \rho_2}\, \sin\delta \quad = J \text{ – current from } 1 \to 2$$

$$\dot{\rho}_2 = -\frac{2}{\hbar} K \sqrt{\rho_1 \rho_2}\, \sin\delta \quad = -J \text{ – current from } 2 \to 1$$

$$\dot{\theta}_1 = \frac{K}{\hbar}\sqrt{\frac{\rho_2}{\rho_1}}\cos\delta - \frac{qV}{2\hbar}$$

$$\dot{\theta}_2 = \frac{K}{\hbar}\sqrt{\frac{\rho_1}{\rho_2}}\cos\delta + \frac{qV}{2\hbar}$$

$$\Rightarrow \quad \boxed{J = J_0 \sin\delta} \qquad J_0 = \frac{2K\rho_0}{\hbar}$$

$\rho_1 = \rho_2 = \rho_0 = \text{const.}$ due to a battery

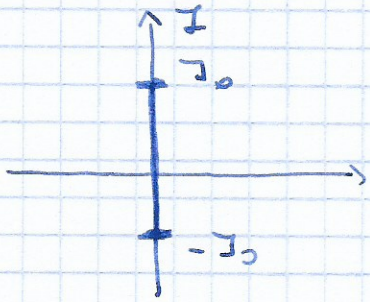
$$\dot{\delta} = \dot{\theta}_2 - \dot{\theta}_1 = \frac{eV}{\hbar}$$

$$\boxed{\delta(t) = \delta_0 + \frac{e}{\hbar}\int V(t)\,dt}$$

$$\underline{J(t) = J_0 \sin(\delta(t))}$$

DC for a constant $V = V_0 \rightarrow \delta(t) = \delta_0 + \frac{e}{\hbar} V_0 t$

↑ small $\hbar$



Anderson, Rowell 1963

at $V_0 = 0$ there is a Josephson current $\underline{J = J_0 \sin\delta_0}$

if $V_0 \neq 0$ the current averages to zero | <u>thermodynamic current</u>

AC for a $V(t) = V_0 + v \cos\omega t$

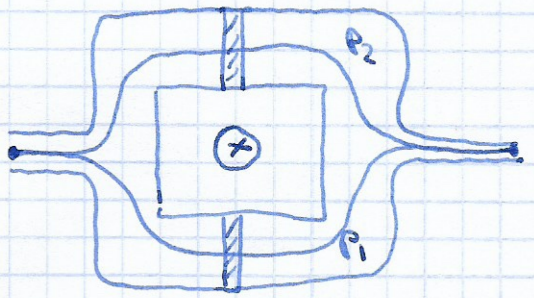
$$\delta(t) = \delta_0 + \frac{e}{\hbar} V_0 t + \frac{e}{\hbar}\frac{v}{\omega}\sin(\omega t)$$

$\sin(x + \Delta x) \simeq \sin x + \Delta x \cos x$

$$J = J_0\Big[\underbrace{\sin(\delta_0 + \tfrac{e}{\hbar} V_0 t)}_{\text{averages to zero}} + \underbrace{\tfrac{e}{\hbar}\tfrac{v}{\omega}\sin(\omega t)\cos(\delta_0 + \tfrac{e}{\hbar} V_0 t)}_{\text{contributes if } \omega = \frac{e}{\hbar} V_0}\Big]$$

Shapiro, 1963

•) <u>consider two Josephson junction</u>



the phase difference

$$\boxed{\delta_2 - \delta_1 = \frac{2e}{\hbar}\oint_C \bar{A}\cdot d\bar{r}}$$

Cooper pair $|k\uparrow, -k\downarrow\rangle$
of charge $Q = 2e$

$$\delta_2 - \delta_1 = \frac{2e}{\hbar}\Phi$$

↑ magnetic flux

Now $\delta_1 = \delta_0 + \frac{e}{\hbar}\Phi$

$\delta_2 = \delta_0 - \frac{e}{\hbar}\Phi$

$$J_{tot} = J_0 \left(\sin\left(\delta_0 + \frac{e}{\hbar}\Phi\right) + \sin\left(\delta_0 - \frac{e}{\hbar}\Phi\right)\right) =$$

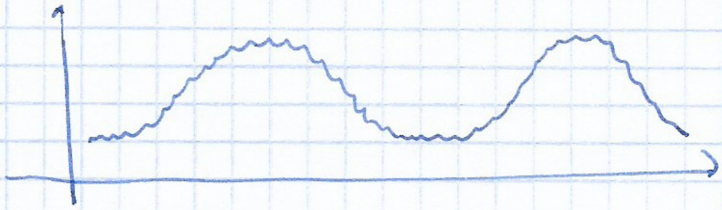
$$= J_0 \sin(\delta_0) \cos\left(\frac{e\Phi}{\hbar}\right)$$

The maximal current is

$$J_{max} = J_0 \left| \cos\left(\frac{e\Phi}{\hbar}\right)\right|$$

and oscillates with $\Phi = \int \overline{B} \cdot d\overline{S}$.

The maxima are at $\Phi = n \frac{\pi \hbar}{e}$, $n \in \mathbb{Z}$



very sensitive device to detect a weak magnetic field → computer tomograph

medicine

SQUID – Superconductor Quantum Interference Device

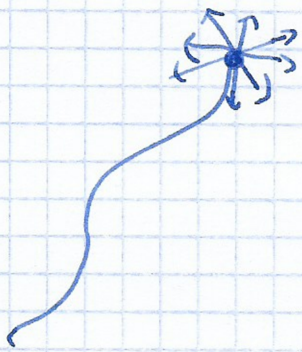
§7. Dirac magnetic monopole

Dirac 1931 – compatibility between existence of a magnetic monopole and quantum mechanics.

Classically a magnetic monopole does not exist!

$$\bar{\nabla} \cdot \bar{B} = 0$$

But, as an end point of a semi-infinite solenoid it can exist



This gives finite $\bar{A}$ in QM and Schrödinger equation → $\bar{A}$ - observable(?)

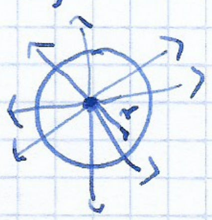
Dirac found the way out

$\boxed{\text{Dirac veto}}$

Let in $\mathbb{R}^3$ $\boxed{\bar{B} = \frac{g}{r^2}\hat{e}_r}$ – radial magnetic field

Solution of $\boxed{\bar{\nabla} \cdot \bar{B} = 4\pi g\, \delta(\bar{r})}$ g – monopole strength

Total magnetic flux $\oint \bar{B} \cdot d\bar{S} = g \int d\Omega\, r^2 \frac{g}{r^2} =$



$= 4\pi g$

consider $\mathbb{R}^3 \setminus \{0\} \simeq S^2 \times \mathbb{R} \sim S^2$

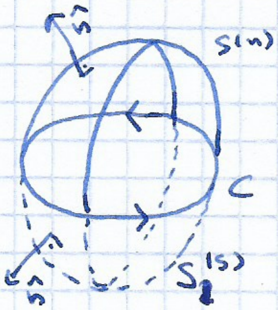
What is $\bar{A}$ associated with such $\bar{B}$?

$$\bar{B} = \bar{\nabla} \times \bar{A}$$

Theorem: No a single regular / smooth vector potential $\bar{A}$ exists on the whole $\mathbb{R}^3 - \{0\}$

d. a. Let $\bar{A}$ exists such that $\oint_C \bar{A}\cdot d\bar{l} = \int_{S^2} \bar{B}\cdot d\bar{S} = 4\pi g$

smooth on the whole S^2 for any closed line C on S^2



on the other hand, applying Stokes' theorem

$$\int_{S = S^{(s)} + S^{(n)}} \bar{B}\cdot d\bar{S} = \oint_C \bar{A}\cdot d\bar{l} - \oint_C \bar{A}\cdot d\bar{l} = 0$$

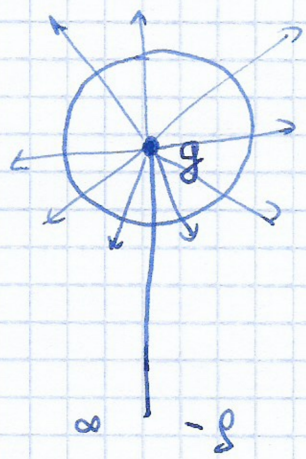
↑ different orientation

in contradiction to $4\pi g \neq 0$

$\square$

$\Rightarrow$ If a monopole exists there should be at least one point on S^2 where $\bar{A}$ is singular.

By connecting such singular points when radius is varied one get a string - Dirac string



The total flux through S^2 would be zero

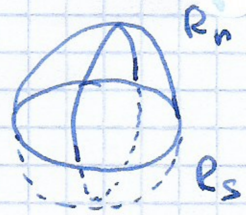
To be consistent with QM Dirac showed the the wave function in a presence of singular $\bar{A}$ must vanish along the string $\rightarrow$ boundary condition $\rightarrow$ $\boxed{4\pi g = n\,\frac{h}{e} \text{ quantized}}$

Dirac veto

1975 T. Wu and C.N. Yang used different (modern) argument, eliminating the necessity of string

We work with two-distinct gauges $\bar{A}^{(n)}$ and $\bar{A}^{(s)}$

well defined in a subset of space R_n and R_s



$R_n + R_s = S^2$

$\bar{A}^{(s)} = \bar{A}^{(n)} + \nabla\chi$

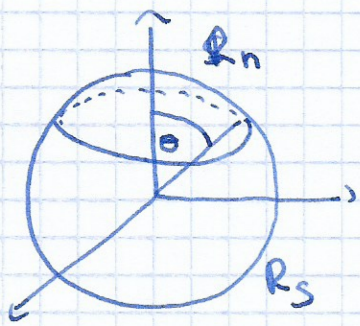
Then, since $\bar{A}^{(n)} \neq \bar{A}^{(s)}$

$$\int_{S^2} \bar{B}\cdot d\bar{S} = \oint_c d\bar{l}\cdot(\bar{A}^{(n)} - \bar{A}^{(s)}) \neq 0$$

To find $\bar{A}^{(n)}$ and $\bar{A}^{(s)}$ consider a simple (parallel) path

$C = C_\theta$

north and south caps:



φ – azimuthal angle

θ – polar angle

$L = \int_0^{2\pi} d\varphi \int_0^{\theta} d\theta' \sin\theta'\, r^2 \frac{g}{r^2} =$

$= -2\pi g \cos\theta' \Big|_0^{\theta} = 2\pi g(1-\cos\theta)$

$R = \int_0^{2\pi} d\varphi\, r\sin\theta\, \bar{A}^{(n)}\cdot\bar{e}_\varphi =$

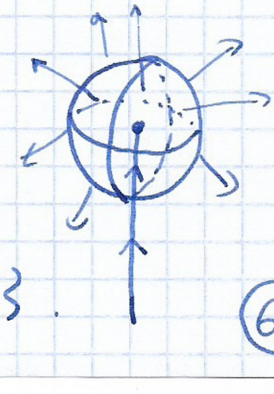
$= 2\pi r \sin\theta\, \bar{A}^{(n)}\cdot\bar{e}_\varphi$

Stokes' theorem on R_n cap

$$\int_N \bar{B}\cdot dS\,\bar{e}_r = 2\pi(1-\cos\theta)g = \int \bar{A}^{(n)}\cdot d\bar{l} =$$

$$= 2\pi r \sin\theta\, \bar{A}^{(n)}\cdot\bar{e}_\varphi$$

$\Rightarrow$ $$\boxed{\bar{A}^{(n)} = \frac{g(1-\cos\theta)}{r\sin\theta}\,\bar{e}_\varphi}$$



singular at $\theta = \pi$ – Oz^- axis.

$\bar{A}^{(n)}$ well defined on $\mathbb{R}^3 - \{Oz^-\}$.

Stokes' theorem on south cap

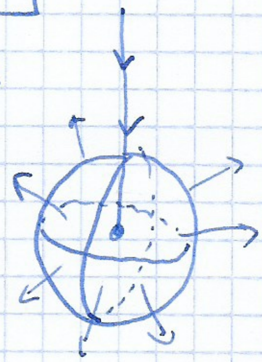
$$\int_S \bar{B} \cdot dS\,\bar{e}_r = 2\pi(1+\cos\theta)g = -\int \bar{A}^{(s)} \cdot d\bar{l} =$$

$$= -2\pi r \sin\theta\, \bar{A}^{(s)} \cdot \bar{e}_\varphi$$

$$\boxed{\bar{A}^{(s)} = -\frac{g(1+\cos\theta)}{r\sin\theta}\,\bar{e}_\varphi}$$

Singular at $\theta = 0$ - Oz^+ axis

$\bar{A}^{(s)}$ well defined on $\mathbb{R}^3 \setminus \{Oz^+\}$



On any parallel line C_θ

$$\bar{A}^{(n)} - \bar{A}^{(s)} = \frac{2g}{r\sin\theta}\,\bar{e}_\varphi = \bar{\nabla}\underbrace{(2g\varphi)}_{\chi}$$

$$\boxed{\bar{\nabla} = \frac{\partial}{\partial r}\bar{e}_r + \frac{1}{r}\frac{\partial}{\partial\theta}\bar{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\bar{e}_\varphi}$$

All of that was classical.

QM enters by observation that the wave functions on C_θ in patches (n) and (s) differ by a phase factor

$$\psi^{(n)}(\bar{r}) = \psi^{(s)}(\bar{r})\, e^{i\frac{e}{\hbar}\int^{\bar{r}}(\bar{A}^{(n)} - \bar{A}^{(s)})\cdot d\bar{l}} =$$

$$= \psi^{(s)}(\bar{r})\, e^{i\frac{e}{\hbar}2g\varphi}$$

azimuthal angle

the wave function must be single valued

$\varphi \leftrightarrow \varphi + 2\pi n \qquad n \in \mathbb{Z}$

hence, $\quad \frac{e}{\hbar} 2g\, 2\pi = 2\pi n \qquad \boxed{e^{i\varphi} = e^{i\varphi + 2\pi n i}}$

Quantization of a monopole magnetic flux

$$\boxed{4\pi g = n \frac{h}{e} = n \Phi_0 \qquad n \in \mathbb{Z}}$$

flux quantum

This is a topological quantization - due to a gluing condition on C_θ not a b.c. in eigen value problem

Dirac → if magnetic monopole exists then its strength is quantized

$$g = \frac{n}{4\pi} \frac{h}{e}$$

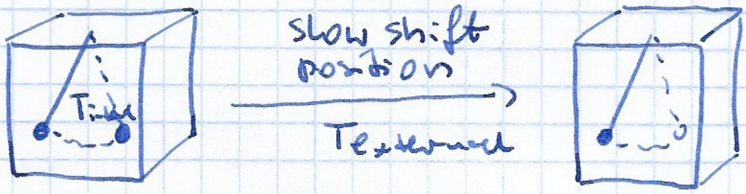
But then it is unvisible

IV BERRY PHASE AND TOPOLOGICAL STATES IN QUANTUM MECHANICS

§ 1. Adiabatic approximation

•) Adiabatic processes

classical pendulum

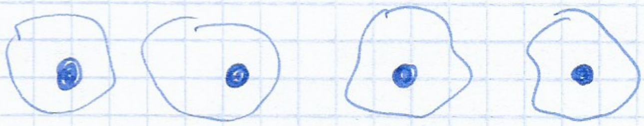


adiabatic process if $\boxed{T_{internal} \ll T_{external}}$

Example $\quad T = 2\pi\sqrt{\frac{L}{g}}$ and change slowly $L = L(t)$

if $\frac{1}{L(t)}\frac{dL(t)}{dt} \ll \frac{1}{T} \quad \longrightarrow \quad \boxed{T(t) = 2\pi\sqrt{\frac{L(t)}{g}}}$

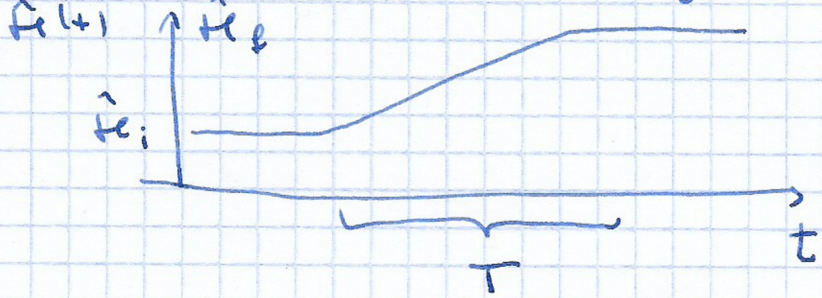
Example $\quad$ Born - Oppenheimer approximation



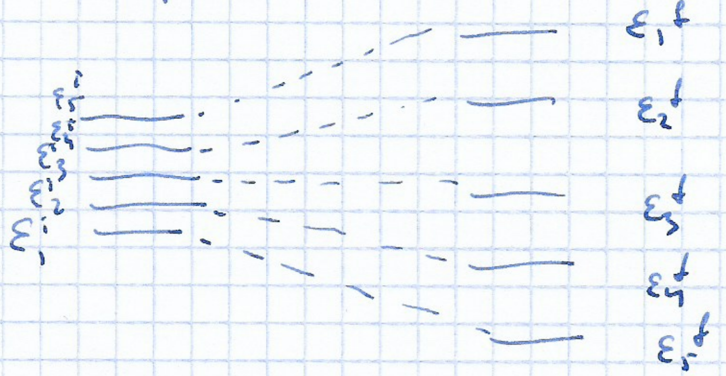
$$\psi(\bar{r}_e, \bar{R}_i) = \psi_{el}(\bar{r}_e)\cdot\psi_i(\bar{R}_i)$$

• Adiabatic theorem in quantum mechanics

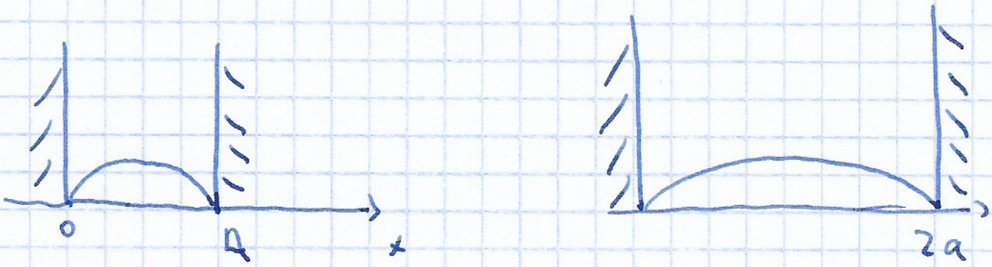
$\hat{H}$ is changed gradually from $\hat{H}_i$ to $\hat{H}_f$



Theorem: If a system was in nth eigenstate of $\hat{H}_i$ then it remains in nth eigenstate of $\hat{H}_f$ [Ehrenfest, Born, Fock - 1928]



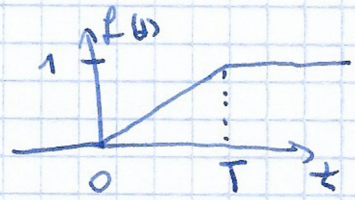
Example



$$\psi_i(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{\pi}{a}x\right) \xrightarrow[\text{slow}]{} \psi_f(x) = e^{i\varphi}\sqrt{\frac{1}{a}} \sin\left(\frac{\pi}{2a}x\right)$$

proof

$$\hat{H}(t) = \underbrace{\hat{V}}_{\text{"small"}} f(t)$$



$$\Psi_n(0) = \underbrace{\psi_n^i}_{\text{eigenstate of } \hat{H}_i} \longrightarrow \Psi(t)$$

$\boxed{\text{states are not degenerate}}$

If $f(t)$ changes slowly

$$|\langle \Psi(T) | \underbrace{\psi_m^f}_{\text{eigenstate of } \hat{H}_f} \rangle|^2 = \begin{cases} 1 & m = n \\ 0 & m \neq n \end{cases}$$

→ 1st order of perturbation without time $\quad V_{km} = \langle \psi_k | \hat{V} | \psi_m \rangle$

$$\boxed{\psi_m^f = \psi_m + \sum_{k \neq m} \frac{V_{km}}{E_m - E_k} \psi_k} \quad (*)$$

states and energies at $t=0$

→ 1st order of perturbation with time

$$\boxed{\Psi(t) = \sum_l c_l(t) \psi_l e^{-i \frac{E_l t}{\hbar}}}$$

$$c_l = \begin{cases} 1 - \frac{i}{\hbar} V_{nn} \int_0^t f(t')\, dt' & l = n \\ -\frac{i}{\hbar} V_{ln} \int_0^t f(t') e^{i(E_l - E_n)t'/\hbar}\, dt' & l \neq n \end{cases}$$

note: $$e^{i \frac{E_l - E_n}{\hbar} t} = -\frac{i\hbar}{E_l - E_n} \frac{d}{dt} e^{i \frac{E_l - E_n}{\hbar} t}$$

and integrates by parts $(l \neq n)$

$$c_l(t) = -\frac{V_{ln}}{E_l - E_n} \int_0^t f(t') \frac{d}{dt'} \left[e^{i \frac{E_l - E_n}{\hbar} t'} \right] dt' =$$

$$= -\frac{V_{ln}}{E_l - E_n} \left[f(t) e^{i \frac{E_l - E_n}{\hbar} t'} \Big|_0^t - \int_0^t \frac{df}{dt'} e^{i \frac{E_l - E_n}{\hbar} t'} dt' \right]$$

$f(0) = 0$

adiabatic condition $\quad T_{ext} \gg T_{int}$

$$\boxed{\frac{1}{T_{ext}} = \frac{1}{f} \frac{df}{dt} \ll \frac{|E_l - E_n|}{\hbar} = \frac{1}{T_{int}}}$$

adiabatic approximation valid if

$$\boxed{\frac{df}{dt} \ll \frac{|E_L - E_n|}{\hbar} f}$$

$$e^{i\frac{E_L}{\hbar}T} \cdot e^{-i\frac{E_L T}{\hbar}} = 1$$

$$(**)\quad \boxed{\Psi(T) = \left[\left(1 - i\frac{V_{nn}A}{\hbar}\right)\psi_n - \sum_{l\neq n}\frac{V_{ln}}{E_l - E_n}\psi_l\right] e^{-i\frac{E_n T}{\hbar}}}$$

$A = \int_0^T f(t)\,dt$ - area of $f(t)$ from 0 to T

$f(T) = 1$

compute the scalar product *) & (**)

$$\langle \Psi(T) | \psi_m^f \rangle = \left[\left(1 + i\frac{V_{nn}^* A}{\hbar}\right)\langle \psi_n| - \sum_{l\neq n}\frac{V_{ln}^*}{E_l - E_n}\langle\psi_l|\right] e^{i\frac{E_n T}{\hbar}} \cdot$$

$$\underbrace{\left[\,|\psi_m\rangle + \sum_{k\neq m}\frac{V_{km}}{E_m - E_k}|\psi_k\rangle\right]}_{O(V_{nm})} =$$

$$= \left[\left(1 + i\frac{V_{nn}^* A}{\hbar}\right)\underbrace{\langle\psi_n|\psi_m\rangle}_{\delta_{nm}} - \sum_{l\neq n}\underbrace{\frac{V_{lm}^*}{E_l - E_n}}_{O(V_{nm})}\underbrace{\langle\psi_l|\psi_m\rangle}_{\delta_{lm}}\right] e^{i\frac{E_n T}{\hbar}} + O(V_{nm}^2)$$

up to $O(1)$ terms

$$\boxed{\langle \Psi(T) | \psi_m^f \rangle = \begin{cases} 1 & n = m \\ 0 & n \neq m \end{cases}}$$

If $\hat{V}$ not "small" we can split $T \to \frac{T}{N}$ such that $\left(\frac{\hat{V}}{N}\right)$ - "small"

$\square$