Tutorials for Advanced Quantum Mechanics of Many Body Systems

winter term 2020-21

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Lecture
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Lectures and general rules

`https://www.fuw.edu.pl/~pjak/Advanced_QM_2020/index.html`

Permanent link for tutorials

Krzysztof Byczuk
`https://zoom.us/j/4899364871?pwd=YWJrQlZ4ems2WWV6R1ZBSERvUy9ZUT09`
Meeting ID: 489 936 4871
Passcode: 3XTpkb

Rules for tutorials

In every week a randomly selected group of students is supposed to prepare a solution to the given problem. A random person from each group will be request to present the solution on-line and me and the rest of you can discuss it and comment. After a satisfactory presentation this group will be given one point (plus, +), which will count in a total classification.

The list of students with their numbers is here. Arranging a work in a given group, please contact each other by yourself. Student's e-mails you can find from Department's WWW page. Perhaps you can make a dedicated group on Facebook or Messenger.

On Mondays and Wednesdays around 12:00 (midday) you can contact me on Zoom to consult your problem. Please earlier arrange this meeting by e-mail. In case of short questions, just write e-mails.

Understanding all problems occurring with on-line teaching, I ask you to contact me as much as you need. I will be happy to help you and assist you to master the topics of this course. Be also creative by yourself.

Tutorial participants

1. Bakhshizada Rashad
2. Basualdo García Siwar José
3. Błaszkiewicz Rafał
4. Chełstowski Igor
5. Cieśliński Dominik Sebastian
6. Ćwiek Rafał Ernest
7. Deka Anjan
8. Dilcher Klaudia Hanna
9. Gajewska Joanna
10. Girguś Mariusz
11. Hafiz Afwan
12. Kalužná Zlatica
13. Karny Michał Marek
14. Kołodziejczyk Jan Mateusz
15. Kotek Aleksandra Małgorzata
16. Kuczński Bartosz
17. Łukanowski Karol
18. Nowak Kacper
19. Nuszkiewicz Antoni Robert
20. Pawlak Jakub Paweł
21. Poniatowski Mateusz Zdzisław
22. Pruszczyk Marcin Piotr
23. Suchorowski Michał Wojciech
24. Suwała Dominik Kamil
25. Turowski Mateusz Jan
26. Walewski Maks Zachary
27. Wojciechowska Agata
28. Wrzosek Piotr

If you are not on this list please contact me ASAP.

1 Week I, 15-21/10/2020

1.1 Tutorial

1. *What are dimensions of bra and ket objects?* - Following the paper arXiv:2008.03187 we perform a dimensional analysis of $|\Psi\rangle$, $|\vec{r}\rangle$, $|\vec{p}\rangle$, and $|n\rangle$ vectors. We discuss basic principles and definitions in quantum mechanics.

2. *Harmonic oscillator, algebraic solution* - We solve a quantum harmonic oscillator in one dimension by introducing creation and annihilation operators and using their commutation algebra we find algebraically the spectrum of this problem.

2 Week II, 22-28/10/2020

2.1 Tutorial

1. *Spin singlet* - Consider two spins 1/2 in a singlet state
$$|\Psi\rangle = \frac{1}{\sqrt{2}}[|\uparrow\rangle_A \otimes |\downarrow\rangle_B - |\downarrow\rangle_A \otimes |\uparrow\rangle_B].$$
a) What is probability to measure a spin $+\hbar/2$ on the subsystem A if no measurement was performed on the subsystem B?
b) If the measurement on the spin B was $+\hbar/2$ what is a result of the subsequent measurement on A?

   [Solution will be prepared and presented by students 9-16.]

2. *Chain of atoms* - A system is made on N lattice sites (ions) distributed uniformly around a circle. This is a simple model of one dimensional crystal. An electron can be localized around one of the site having the energy E_0 and can hop between nearest neighbor sites with the probability amplitude W. The Hamiltonian of the system is
$$\hat{H} = \sum_{n=1}^{N} E_0|n\rangle\langle n| + \sum_{n=1}^{N} W(|n\rangle\langle n+1| + |n+1\rangle\langle n|),$$
where $\{|n\rangle\}$ is a complete and orthonormal base and $n = 1, 2, ..., N$. Assuming a periodic boundary condition, i.e. $|N\rangle = |1\rangle$ find the spectrum of the Hamiltonian.

   Hint: Introduce an operator $\hat{A} = \sum_{n=1}^{N} |n\rangle\langle n+1|$ and its hermitian conjugation and express the Hamiltonian by them. Note that $\hat{A}^\dagger\hat{A} = 1$ and check how they act on a state $|n\rangle$. Write down the operator A as a matrix in the base $\{|n\rangle\}$. Find the spectrum of $\hat{A}$.

   [Solution will be prepared and presented by students 1-8.]

3. *Factorization method* - Following Infeld and Hull method (see L. Infeld and T.E. Hull Rev. Mod. Phys **23**, 21 (1951) and H.C. Ohanian, *Principles of quantum mechanics*) we discuss the way how to factorize a solvable Hamiltonian and how to find its spectrum in an algebraic manner. We introduce a set of ladder operators $\hat{\eta}_i$ and real constants E_i such that
$$\hat{\eta}_1^\dagger\hat{\eta}_1 + E_1 = \hat{H}$$
$$\hat{\eta}_{j+1}^\dagger\hat{\eta}_{j+1} + E_{j+1} = \hat{\eta}_j\hat{\eta}_j^\dagger + E_j$$
where $j = 1, 2, 3...$ and $\hat{H}$ is the Hamiltonian of the system, which discrete spectrum we are looking for.
Theorem: Suppose that these equations are satisfied and that $\hat{\eta}_i$ has an eigenvector $|\xi_i\rangle$ with eigenvalue zero, i.e. $\hat{\eta}|\xi_i\rangle = 0$. Then, a) the constants E_j is the jth eigenvalue of $\hat{H}$ (arrange is ascending order), b) the corresponding eigenvector (up to normalization) is
$$|E_j\rangle = \hat{\eta}_1^\dagger\hat{\eta}_2^\dagger\hat{\eta}_3^\dagger....\hat{\eta}_{j-1}^\dagger|\xi_j\rangle.$$
Prove this theorem. Hints: Introduce an operator $\hat{\Lambda} = \hat{\eta}_j^\dagger\hat{\eta}_j + E_j$ and show that
$$\hat{\Lambda}_{j+1} = \hat{\eta}_j\hat{\eta}_j^\dagger + E_j,$$
$$\hat{\Lambda}_{j+1}\hat{\eta}_j = \hat{\eta}_j\hat{\Lambda}_j,$$
$$\hat{\Lambda}_j\hat{\eta}_j^\dagger = \hat{\eta}_j^\dagger\hat{\Lambda}_{j+1}.$$
Since $\hat{H} = \hat{\Lambda}_1$ show that $\hat{H}|E_j\rangle = E_j|E_j\rangle$. Checking the difference $E_{j+1} - E_j$ show that $E_1 \leqslant E_2 \leqslant E_3 \leqslant ....$ Finally, show that if E is an eigenvalue then it must be one of E_j or must be above some maximal (bound) $E_n^{\max}$.

   [Solution will be prepared and presented by students 17-25.]

4. *Algebraic solution of the infinite quantum well* - Using a factorization method find eigenvalues and eigenfunctions of a quantum particle in one dimensional infinite quantum well. Hint: make an Ansatz $\hat{\eta}_j = \frac{1}{\sqrt{2m}}(\hat{p} + if_j(x))$.

   [Solution: depending on time, I will present the solution or ask you to do this in the week III.]

3 Week III, 29/10-04/11/2020

3.1 Tutorial

1. *(Anti)Symmetrization operator as a projector* - Show that (anti)symmetrization operator
$$\mathcal{P}_{B,F}\Psi(\vec{r}_1, ..., \vec{r}_N) = \frac{1}{N!}\sum_P \zeta^P\Psi(\vec{r}_{p_1}, ..., \vec{r}_{p_N}),$$
where $\zeta = \pm$ for bosons and fermions, respectively, is a projection operator.

   [Solution will be prepared and presented by students 1, 5, 14, 19, 22, 25.]

2. *Interacting two bosons in a harmonic potential* - Two identical bosons moving in one dimension in a harmonic potential $U(x) = \frac{1}{2}m\omega^2 x^2$ interact with each other by the potential $V(x_1, x_2) = \alpha\exp(-\beta(x_1 - x_2)^2)$, where α is a real constant and β is a real and positive constant. Find the ground state energy in the first order perturbation theory.

   [Solution will be prepared and presented by students 2, 4, 9, 13, 18, 20, 26.]

3. *Two interacting electrons in a quantum well* - Two electrons of mass m and spins $1/2$ are in moving in one dimensional infinite quantum well. They interact with each other with a spin independent potential $V(x_1 - x_2)$. Treating this potential as a perturbation describe the three lowest energy states in terms of one particle states of the quantum well and the spins of particles. Determine in the first order of perturbation the ground state energy and the energy of the first two excited states. Hints: results express in integral forms.

   [Solution will be prepared and presented by students 6, 8, 10, 12, 15, 24, 27.]

4 Week IV, 04-11/11/2020

4.1 Tutorial

1. *One and two particle density matrices* - A many body wave function of electrons contains a huge number of information which is wholly almost irrelevant. Therefore, it is advantageous to introduce a one particle density matrix and a two particle density matrix as follows

$$\gamma_\sigma(\vec{r}_1'\sigma_1', \vec{r}_1\sigma_1) = N \sum_{\sigma_2,\dots,\sigma_N} \int dr_2 \dots dr_N$$

$$\cdot \Psi^*(\vec{r}_1'\sigma_1', \vec{r}_2\sigma_2, \dots, \vec{r}_N\sigma_N)\Psi(\vec{r}_1\sigma_1, \vec{r}_2\sigma_2, \dots, \vec{r}_N\sigma_N)$$

   and

$$\Gamma_\sigma(\vec{r}_1'\sigma_1', \vec{r}_2'\sigma_2'; \vec{r}_1\sigma_1, \vec{r}_2\sigma_2) = \frac{N(N-1)}{2}$$

$$\cdot \sum_{\sigma_3,\dots,\sigma_N} \int dr_3 \dots dr_N \Psi^*(\vec{r}_1'\sigma_1', \vec{r}_2'\sigma_2', \vec{r}_3\sigma_3, \dots, \vec{r}_N\sigma_N)$$

$$\cdot \Psi(\vec{r}_1\sigma_1, \vec{r}_2\sigma_2, \vec{r}_3\sigma_3, \dots, \vec{r}_N\sigma_N),$$

   and their spinless forms

$$\gamma(\vec{r}_1', \vec{r}_1) = \sum_{\sigma_1} \gamma_\sigma(\vec{r}_1'\sigma_1, \vec{r}_1\sigma_1),$$

$$\Gamma(\vec{r}_1', \vec{r}_2'; \vec{r}_1, \vec{r}_2) \sum_{\sigma_1\sigma_2} \Gamma_\sigma(\vec{r}_1'\sigma_1, \vec{r}_2'\sigma_2; \vec{r}_1\sigma_1, \vec{r}_2\sigma_2).$$

   Express the total energy of the system in a given state

$$E = \sum_{\text{spin}} \int dr_1 \dots dr_N \Psi^* \hat{H} \Psi,$$

   where

$$\hat{H} = \sum_{i=1}^{N} (-\frac{\hbar^2}{2m}\nabla_i + U(\vec{r}_i)) + \sum_{i<j=1}^{N} V(\vec{r}_i, \vec{r}_j)$$

   by these density matrices.

   [Solution will be prepared and presented by students 2, 6, 11, 18, 22, 24.]

2. *Normalization of symmetric states* - Show that for a symmetric state $|\alpha_1, \dots, \alpha_N\}$ the scalar product is

$$\{\alpha_1', \dots \alpha_N' | \alpha_1, \dots, \alpha_N\} = \zeta^P \prod_\alpha (n_\alpha!),$$

   where $(\alpha_1', \dots, \alpha_N')$ is a permutation of $(\alpha_1, \dots, \alpha_N)$ and this scalar product is zero otherwise. n_α is a number of particles in the one-particle state α.

   [Solution will be prepared and presented by students 3, 9, 17, 20, 23, 26, 27.]

3. *Bosons in a box* - Three spinless bosons are in a one-dimensional box of length L with periodic boundary conditions. The one-particle base is $\{|k\rangle\}$, where $k = 2\pi n/L$ ($n = 0, \pm1, \pm2, ..$) and the corresponding wave functions are $\psi_k(x) = \langle x|k\rangle = \exp(ikx)/\sqrt{L}$. The three bosons are in one-particle states with $k_1 = 0$ and $k_2 = k_3 = 2\pi/L$.
   a) Express explicitly the states $|k_1, k_2, k_2\}$ and $|k_1, k_2, k_2\rangle$.
   b) Compute $\{k_1, k_2, k_2|k_1, k_2, k_2\}$ and $\langle k_1, k_2, k_2|k_1, k_2, k_2\rangle$.
   c) How does the operator $\hat{a}_k$ with a given k act on a state $|k_1, k_2, k_2\}$?
   d) Compute $\hat{a}_i^\dagger \hat{a}_i |k_1, k_2, k_2\rangle$, for $i = k_1$ and k_2.

   [Solution will be prepared and presented by students 4, 7, 14, 16, 19, 21, 28.]

5 Week V, 12-18/11/2020

5.1 Tutorial

There is no tutorial meeting, official change in the schedule.

6 Week VI, 19-25/11/2020

6.1 Tutorial

1. *Fermions in a box* - Three electrons with spin $\hbar/2$ are in a one-dimensional box of length L with periodic boundary conditions. The one-particle base is $\{|k\rangle\}$, where $k = 2\pi n/L$ ($n = 0, \pm1, \pm2, ..$) and the corresponding wave functions are $\psi_k(x) = \langle x|k\rangle = \exp(ikx)/\sqrt{L}$. Consider a state $|\psi\rangle = |\alpha_1, \alpha_2, \alpha_3\}$, where the electrons are in one-particle states $\alpha_1 = (0, \uparrow)$, $\alpha_2 = (0, \downarrow)$, and $\alpha_3 = (2\pi/L, \uparrow)$, where we write $\alpha_i = (k_i, \sigma_i)$. Find results of acting on state $|\psi\rangle$ with the following operators:

a) $\hat{a}^{\dagger}_{\frac{2\pi}{L},\uparrow}$,
b) $\hat{a}^{\dagger}_{\frac{2\pi}{L},\downarrow}$,
c) $\hat{a}_{0,\uparrow}$,
d) $\hat{a}_{0,\downarrow}$,
e) $\hat{a}_{\frac{2\pi}{L},\uparrow}$,
f) $\hat{a}_{\frac{2\pi}{L},\downarrow}$.

[Solution will be prepared and presented by students 1, 5, 8, 10, 12, 13, 15, 25.]

2. *Second quantized operators* - Express in second quantization formalizm the following operators for N-electron system:

   - The spin operator $\hat{\mathbf{S}} = \sum_{i=1}^{N} \frac{\hbar}{2}\hat{\sigma}_i$ [Solution will be prepared and presented by students 3, 7, 11, 16, 17, 27, 28.]
   - The kinetic energy $\hat{T} = \sum_{i=1}^{N} \frac{\hat{\mathbf{p}}_i^2}{2m}$ (in position representation) [Solution will be prepared and presented by students 4, 6, 14, 18, 22, 23, 24.]
   - The particle density operator $\hat{n}(\mathbf{r}) = \sum_{i=1}^{N} \delta(\mathbf{r}-\mathbf{r}_i)$ (in momentum representation) [Solution will be prepared and presented by students 2, 9, 19, 20 21, 26.]

3. *Quantization of vibrating string* - Quantize a one dimensional string that can vibrate. Compare this with a quantized one dimensional chain of atoms. [Presented by KB.]

7 Week VII, 26/11-02/12/2020

7.1 Tutorial

1. *Quantization of vibrating string* - Quantize a one dimensional string that can vibrate. Compare this with a quantized one dimensional chain of atoms. [Presented by KB.]

8 Week VIII, 03-09/12/2020

8.1 Tutorial

1. *Gas of ideal fermions* - Discuss properties of the ground state of a gas of ideal fermions. [Presented by KB.]

2. *Pauli paramagnetism* -Discuss the ground state of ideal fermions of spin one-half in a uniform magnetic field. Find the Pauli magnetic susceptibility. [Presented by KB.]

9 Week IX, 10-16/12/2020

9.1 Tutorial

1. *Casimir effect* - Discuss the Casimir problem of two parallel metalic plates in the electromagnetic field. Find the force between these two plates. [Presented by KB.]