§ BCS theory

•) BCS mean-field theory

$$H = \sum_{k\sigma} \xi_k c_{k\sigma}^\dagger c_{k\sigma} + \frac{1}{N}\sum_{kk'} V_{kk'} c_{k\uparrow}^\dagger c_{-k\downarrow}^\dagger c_{-k'\downarrow} c_{k'\uparrow}$$

mean field approximation

$$0 = (\hat{A} - \langle\hat{A}\rangle)(\hat{B} - \langle\hat{B}\rangle) = \hat{A}\hat{B} - \hat{A}\langle\hat{B}\rangle - \hat{B}\langle\hat{A}\rangle + \langle\hat{A}\rangle\langle\hat{B}\rangle$$

$$\Rightarrow \quad \hat{A}\hat{B} \simeq \hat{A}\langle\hat{B}\rangle + \hat{B}\langle\hat{A}\rangle - \langle\hat{A}\rangle\langle\hat{B}\rangle$$

in BCS they $\hat{A} = c_{k\uparrow}^\dagger c_{-k\downarrow}^\dagger$, $\hat{B} = c_{-k\downarrow} c_{k\uparrow}$

$$H_{BCS} = \sum_{k\sigma} \xi_k c_{k\sigma}^\dagger c_{k\sigma} + \frac{1}{N}\sum_{kk'} V_{kk'}\Big[\langle c_{k\uparrow}^\dagger c_{-k\downarrow}^\dagger\rangle c_{-k'\downarrow} c_{k'\uparrow} +$$
$$+ c_{k\uparrow}^\dagger c_{-k\downarrow}^\dagger \langle c_{-k'\downarrow} c_{k'\uparrow}\rangle - \langle c_{k\uparrow}^\dagger c_{-k\downarrow}^\dagger\rangle\langle c_{-k'\downarrow} c_{k'\uparrow}\rangle\Big]$$

let define $\quad \Delta_k = -\frac{1}{N}\sum_{k'} V_{kk'} \langle c_{-k'\downarrow} c_{k'\uparrow}\rangle$

$$\Delta_k^* = -\frac{1}{N}\sum_{k'} V_{kk'} \langle c_{k'\uparrow}^\dagger c_{-k'\downarrow}^\dagger\rangle$$

$$H_{BCS} = \sum_{k\sigma} \xi_k c_{k\sigma}^\dagger c_{k\sigma} - \sum_k \Delta_k^* c_{-k\downarrow} c_{k\uparrow} - \sum_k \Delta_k c_{k\uparrow}^\dagger c_{-k\downarrow}^\dagger +$$
$$+ \text{cont.}$$

•) Bogoliubov-Valatin transformation

$$\begin{pmatrix} \gamma_{k\uparrow} \\ \gamma_{-k\downarrow}^\dagger \end{pmatrix} = \begin{pmatrix} u_k^* & -v_k \\ v_k^* & u_k \end{pmatrix} \begin{pmatrix} c_{k\uparrow} \\ c_{-k\downarrow}^\dagger \end{pmatrix}$$

$$\{\gamma_{k\uparrow}, \gamma_{k\uparrow}^\dagger\} = u_k^* u_k c_{k\uparrow} c_{k\uparrow}^\dagger - u_k^* v_k^* c_{k\uparrow} c_{-k\downarrow} - v_k u_k c_{-k\downarrow}^\dagger c_{k\uparrow}^\dagger + v_k v_k^* c_{-k\downarrow}^\dagger c_{-k\downarrow}$$
$$+ u_k u_k^* c_{k\uparrow}^\dagger c_{k\uparrow} - u_k v_k c_{k\uparrow}^\dagger c_{-k\downarrow}^\dagger - v_k^* u_k^* c_{-k\downarrow} c_{k\uparrow} + v_k^* v_k c_{-k\downarrow} c_{-k\downarrow}^\dagger =$$
$$= |u_k|^2\{c_{k\uparrow}, c_{k\uparrow}^\dagger\} - u_k^* v_k^* \{c_{k\uparrow}, c_{-k\downarrow}\} - v_k u_k\{c_{-k\downarrow}^\dagger, c_{k\uparrow}^\dagger\} + |v_k|^2\{c_{-k\downarrow}^\dagger, c_{-k\downarrow}\} =$$
$$= |u_k|^2 + |v_k|^2 = 1$$

with this constrain the inverse transformation

$$\begin{pmatrix} c_{k\uparrow} \\ c^{+}_{-k\downarrow} \end{pmatrix} = \begin{pmatrix} u_k & v_k \\ -v_k^* & u_k^* \end{pmatrix} \begin{pmatrix} \gamma_{k\uparrow} \\ \gamma^{+}_{-k\downarrow} \end{pmatrix}$$

$$H_{BCS} = \sum_k \{ \xi_k (u_k^* \gamma^{+}_{k\uparrow} + v_k^* \gamma_{-k\downarrow})(u_k \gamma_{k\uparrow} + v_k \gamma^{+}_{-k\downarrow}) +$$

$$+ \xi_k (-v_k \gamma_{k\uparrow} + u_k \gamma^{+}_{-k\downarrow})(-v_k^* \gamma^{+}_{k\uparrow} + u_k^* \gamma_{-k\downarrow}) -$$

$$- \Delta_k^* (-v_k \gamma^{+}_{k\uparrow} + u_k \gamma_{-k\downarrow})(u_k \gamma_{k\uparrow} + v_k \gamma^{+}_{-k\downarrow}) -$$

$$- \Delta_k (u_k^* \gamma^{+}_{k\uparrow} + v_k^* \gamma_{-k\downarrow})(-v_k^* \gamma_{k\uparrow} + u_k^* \gamma^{+}_{-k\downarrow}) \} =$$

$$= \sum_k \{ (\xi_k |u_k|^2 - \xi_k |v_k|^2 + \Delta_k^* v_k u_k + \Delta_k u_k^* v_k^*) \gamma^{+}_{k\uparrow} \gamma_{k\uparrow}$$

$$+ (-\xi_k |v_k|^2 + \xi_k |u_k|^2 + \Delta_k^* u_k v_k + \Delta_k v_k^* u_k^*) \gamma^{+}_{-k\downarrow} \gamma_{-k\downarrow} +$$

$$+ (\xi_k u_k^* v_k + \xi_k u_k^* v_k + \Delta_k^* v_k^2 - \Delta_k u_k^{*2}) \gamma^{+}_{k\uparrow} \gamma^{+}_{-k\downarrow} +$$

$$+ (\xi_k v_k^* u_k + \xi_k v_k^* u_k - \Delta_k^* u_k^2 + \Delta_k v_k^{*2}) \gamma_{-k\downarrow} \gamma_{k\uparrow} \} + \text{const.}$$

terms with $\gamma\gamma$ and $\gamma^+\gamma^+$ must vanish

$$\boxed{2 \xi_k u_k^* v_k + \Delta_k^* v_k^2 - \Delta_k u_k^{*2} = 0}$$

let

$$\Delta_k = |\Delta_k| e^{i\phi_k}$$

$$u_k = |u_k| e^{i\alpha_k}$$

$$v_k = |v_k| e^{i\beta_k}$$

$$0 = 2\xi_k |u_k||v_k| e^{i(\beta_k - \alpha_k)} + |\Delta_k| (|v_k|^2 e^{i(2\beta_k - \phi_k)} - |u_k| e^{i(\phi_k - 2\alpha_k)})$$

special solution (we do not require the general solution)

$$\alpha_k = 0$$
$$\beta_k = \Phi_k$$

$$2\xi_k |u_k||v_k| + |\Delta_k|\left(|v_k|^2 - |u_k|^2\right) = 0$$

Hence

$$4\xi_k^2 |u_k|^2 |v_k|^2 = |\Delta_k|^2\left(|v_k|^4 - 2|u_k|^2|v_k|^2 + |u_k|^4\right)$$

$$\Rightarrow \quad 4\left(\xi_k^2 + |\Delta_k|^2\right)|u_k|^2|v_k|^2 = |\Delta_k|^2\left(|v_k|^4 + 2|u_k|^2|v_k|^2 + |u_k|^4\right) =$$

$$= |\Delta_k|^2\left(|u_k|^2 + |v_k|^2\right)^2 = |\Delta_k|^2$$

$$\Rightarrow \quad |u_k||v_k| = \frac{|\Delta_k|}{2\sqrt{\xi_k^2 + |\Delta_k|^2}}$$

and
$$|u_k|^2 - |v_k|^2 = \frac{2\xi_k |u_k||v_k|}{|\Delta_k|} = \frac{\xi_k}{\sqrt{\xi_k^2 + |\Delta_k|^2}}$$

with $|u_k|^2 + |v_k|^2 = 1$ we get

$$\boxed{\begin{aligned} |u_k|^2 &= \frac{1}{2}\left(1 + \frac{\xi_k}{\sqrt{\xi_k^2 + |\Delta_k|^2}}\right) \\ |v_k|^2 &= \frac{1}{2}\left(1 - \frac{\xi_k}{\sqrt{\xi_k^2 + |\Delta_k|^2}}\right) \\ |u_k||v_k| &= \frac{|\Delta_k|}{2\sqrt{\xi_k^2 + |\Delta_k|^2}} \end{aligned}}$$

•) BCS Hamiltonian and excitation spectrum

$$H_{BCS} = \sum_k \left(\frac{\xi_k^2}{\sqrt{\xi_k^2+|\Delta_k|^2}} + \frac{|\Delta_k|^2}{\sqrt{\xi_k^2+|\Delta_k|^2}} \right) \overbrace{\left(\gamma_{k\uparrow}^\dagger \gamma_{k\uparrow} + \gamma_{-k\downarrow}^\dagger \gamma_{-k\downarrow} \right)}^{+\text{ cnst.}} =$$

$$= \sum_k \sqrt{\xi_k^2+|\Delta_k|^2} \left(\gamma_{k\uparrow}^\dagger \gamma_{k\uparrow} + \gamma_{-k\downarrow}^\dagger \gamma_{-k\downarrow} \right) + \text{cnst}$$

Using that $\xi_{-k} = \xi_k$, $|\Delta_k| = |\Delta_{-k}|$ (parity)

we find

$$\boxed{\begin{array}{c} H_{BCS} \sum_{k\sigma} E_k \gamma_{k\sigma}^\dagger \gamma_{k\sigma} \\ E_k = \sqrt{\xi_k^2 + |\Delta_k|^2} \end{array}}$$

PTO → 4A and 4B

New dispersion relation for fermions!
New quasiparticles !!!

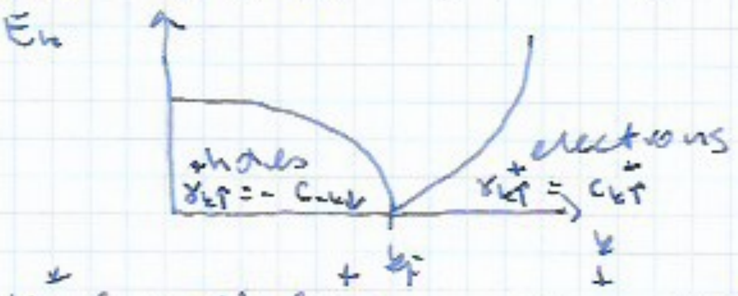
•) Normal $\Delta_k \to 0$ state

$\xi_k := \varepsilon_k - \mu$

$$|u_k|^2 = \frac{1}{2}\left(1 + \frac{\xi_k}{|\xi_k|}\right) = \begin{cases} 0 & \xi_k < 0 \\ 1 & \xi_k > 0 \end{cases}$$

$$\bar{n}_k = |v_k|^2 = \frac{1}{2}\left(1 - \frac{\xi_k}{|\xi_k|}\right) = \begin{cases} 1 & \xi_k < 0 \\ 0 & \xi_k > 0 \end{cases}$$

Bogoliubov quasiparticles described by γ and $\gamma^\dagger$, are holes for energies below ε_F ($\xi_k < 0$) and electrons for energies above ε_F ($\xi_k > 0$). Their dispersion is $E_k = |\xi_k|$



$E_k \geq 0$

$\gamma_{k\uparrow} = u_k^* c_{k\uparrow} - v_k c_{-k\downarrow}^\dagger$, $\gamma_{-k\downarrow}^\dagger = v_k^* c_{k\uparrow} + u_k c_{-k\downarrow}^\dagger$

$\boxed{\gamma_{k\uparrow}^\dagger = u_k c_{k\uparrow}^\dagger - v_k^* c_{-k\downarrow}}$ $\gamma_{-k\downarrow} = v_k c_{k\uparrow}^\dagger + u_k^* c_{-k\downarrow}$

How to define a quasi particle (particle) in QM?

Let $|\Psi\rangle$ a ground state with $\hat{H}^+ = \hat{H}$

$$\hat{H}|\Psi\rangle = E_0|\Psi\rangle \qquad (*)$$

Let $\hat{B}^+|\Psi\rangle \neq 0$ is an excited state such that

$$\hat{H}\hat{B}^+|\Psi\rangle = (E_0 + \varepsilon)\hat{B}^+|\Psi\rangle \qquad (**)$$

Now from $(*)$ / $\hat{B}^+$ and $(**)$

$$\hat{B}^+\hat{H}|\Psi\rangle - \hat{H}\hat{B}^+|\Psi\rangle = -\varepsilon\hat{B}^+|\Psi\rangle$$

$\Rightarrow$ $\boxed{[\hat{B}^+, \hat{H}] = -\varepsilon\hat{B}^+}$ – a proper excitation / quasiparticle with energy ε

Equation of motion

$$i\hbar\frac{d\hat{B}^+}{dt} = [\hat{H}, \hat{B}^+] = \varepsilon\hat{B}^+$$

$\hat{B}^+(t) = e^{i\hat{H}t}\hat{B}^+ e^{-i\hat{H}t}$

$\hookrightarrow$ $\hat{B}^+(t) = \hat{B}^+(0)\, e^{-i\varepsilon t/\hbar}$

$$|\varphi(t)\rangle = e^{-i\varepsilon t/\hbar}\hat{B}^+|\Psi\rangle$$

[stationary $(\tau = \infty)$ quasi particle state.

* * *

For other quantum numbers (energy, charge, spin, ...)

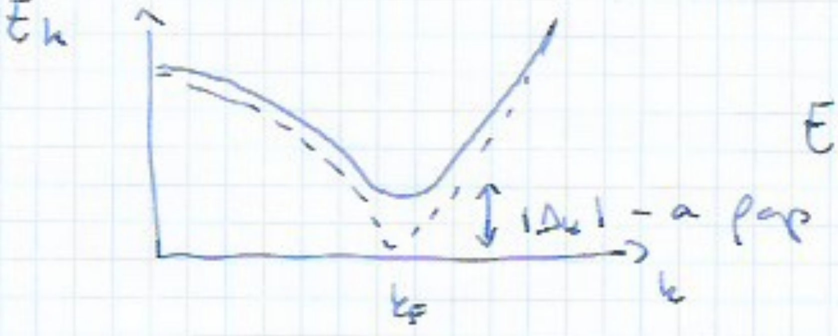
$$\hat{Q}|\Psi\rangle = q|\Psi\rangle$$
$$\hat{Q}\hat{B}^+|\Psi\rangle = (q + \Delta q)\hat{B}^+|\Psi\rangle$$

$\rightarrow$ $\boxed{[\hat{Q}, \hat{B}^+] = \Delta q\,\hat{B}^+}$

$\hat{B}^+$ – a quasiparticle creation operator changes a quantum number $\hat{Q}$ by Δq.

•) superconducting $\Delta_k \neq 0$ state

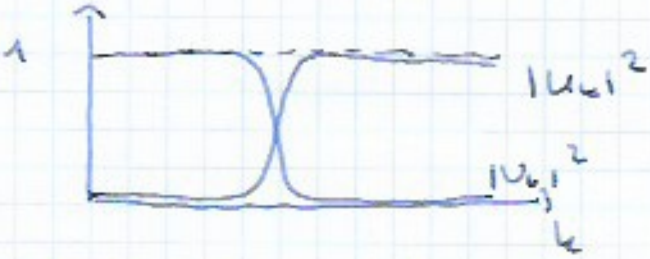
a gap $|\Delta_k|$ in the energy spectrum is present



$$E_k = \sqrt{\xi_k^2 + |\Delta_k|^2}$$

the coherence u_k and v_k factors

$0 < |u_k|^2 < 1$, $\quad 0 < |v_k|^2 < 1$



Bogoliubov quasiparticles are superpositions of electrons and holes. For ~~BdG~~ $k \ll k_F$ they are mostly hole-like. For $k \gg k_F$ they are mostly electron-like. Right at the Fermi level

$$\gamma_{k\uparrow} = \frac{1}{\sqrt{2}} c_{k\uparrow} - \frac{1}{\sqrt{2}} e^{i\phi_k} c^{+}_{-k\downarrow}$$

a superposition of a electron ($c_{k\uparrow}$) and a hole ($c^{+}_{-k\downarrow}$) with the same amplitude. It's charge must be neutral.

$$\gamma^{+}_{k\uparrow} = \underbrace{u_k c^{+}_{k\uparrow}}_{\text{electron}} - \underbrace{v^{*}_k c_{-k\downarrow}}_{\text{hole}}$$

Eigenoperators

$$[\hat{H}, \hat{\gamma}^{+}_{k\uparrow}] = \sum_{k'} E_{k'} [\hat{\gamma}^{+}_{k'\uparrow} \gamma_{k'\uparrow}, \hat{\gamma}^{+}_{k\uparrow}] =$$

$$= \sum_{k'} E_{k'} \left[\gamma^{+}_{k'\uparrow} \underbrace{\gamma_{k'\uparrow} \gamma^{+}_{k\uparrow}}_{\delta_{kk'} - \gamma^{+}_{k\uparrow}\gamma_{k'\uparrow}} + \gamma^{+}_{k\uparrow} \gamma^{+}_{k'\uparrow} \gamma_{k'\uparrow} \right] =$$

$$= \sum_{k'} E_{k'} \left(\delta_{kk'} \gamma^{+}_{k'\uparrow} - \gamma^{+}_{k'\uparrow} \gamma^{+}_{k\uparrow} \gamma_{k'\uparrow} + \gamma^{+}_{k\uparrow} \gamma^{+}_{k'\uparrow} \gamma_{k'\uparrow} \right) =$$

$$= E_k \gamma^{+}_{k\uparrow}$$

$|\Psi_{BCS}\rangle$ - ground state $= \prod_k (u_k + v_k c^{+}_{k\uparrow} c^{+}_{-k\downarrow}) |0\rangle$

electron - excitation

$$\underline{c^{+}_{k\uparrow} |\Psi_{BCS}\rangle} = c^{+}_{k\uparrow} \prod_{k'} (u_{k'} + v_{k'} c^{+}_{k'\uparrow} c^{+}_{-k'\downarrow}) |0\rangle =$$

$$= u_k c^{+}_{k\uparrow} \prod_{k' \neq k} (u_{k'} + v_{k'} c^{+}_{k'\uparrow} c^{+}_{-k'\downarrow}) |0\rangle \equiv$$

$$\equiv \underline{u_k |\Psi_{k\uparrow}\rangle}$$

hole-like excitation

$$\underline{c_{-k\downarrow} |\Psi_{BCS}\rangle} = c_{-k\downarrow} \prod_{k'} (u_{k'} + v_{k'} c^{+}_{k'\uparrow} c^{+}_{-k'\downarrow}) |0\rangle =$$

$$= v_k \underbrace{c_{-k\downarrow} c^{+}_{k\uparrow}}_{\delta - c^{+} c} \underbrace{c^{+}_{-k\downarrow}}_{1 - c^{+} c} \prod_{k' \neq k} (u_{k'} + v_{k'} c^{+}_{k'\uparrow} c^{+}_{-k'\downarrow}) |0\rangle$$

$$= -v_k c^{+}_{k\uparrow} c_{-k\downarrow} c^{+}_{-k\downarrow} \prod_{k' \neq k} (u_{k'} + v_{k'} c^{+}_{k'\uparrow} c^{+}_{-k'\downarrow}) |0\rangle =$$

$$= \underline{-v_k |\Psi_{k\uparrow}\rangle}$$

$c^{+}_{k\uparrow}$ and $c_{-k\downarrow}$ are not eigenoperators since the states are not normalized if $|\Delta_k| \neq 0$.

4B

Bogoliubov excitation

$$\boxed{\gamma_{k\uparrow}^{+} |\Psi_{BCS}\rangle} = \left(u_k c_{k\uparrow}^{+} - v_k^{*} c_{-k\downarrow}\right) |\Psi_{BCS}\rangle =$$

$$= \left(|u_k|^2 + |v_k|^2\right) |\Psi_{k\uparrow}\rangle = \boxed{|\Psi_{k\uparrow}\rangle}$$

normalized excited state

Note that $|\Psi_{BCS}\rangle$ is vacuum state for $\gamma_{k\uparrow}$

$$\boxed{\gamma_{k\uparrow} |\Psi_{BCS}\rangle} = \left(u_k^{*} c_{k\uparrow} - v_k c_{-k\downarrow}^{+}\right) \prod_{k'} \left(u_{k'}^{*} + v_{k'} c_{k'\uparrow}^{+} c_{-k'\downarrow}^{+}\right) |0\rangle$$

$c_{k\uparrow}^{+} c_{k\uparrow} + 1$

$$= u_k^{*} v_k \, c_{k\uparrow} c_{k\uparrow}^{+} c_{-k\downarrow}^{+} \prod_{k' \neq k} \left(u_{k'}^{*} + v_{k'} c_{k'\uparrow}^{+} c_{-k'\downarrow}^{+}\right) |0\rangle -$$

$$- v_k u_k^{*} c_{-k\downarrow}^{+} \prod_{k' \neq k} (\dots) |0\rangle =$$

$$= u_k v_k \, c_{-k\downarrow}^{+} \prod_{k' \neq k} \left(u_{k'} + v_{k'} c_{k'\uparrow}^{+} c_{-k'\downarrow}^{+}\right) |0\rangle -$$

$$- v_k u_k^{*} c_{-k\downarrow}^{+} \prod_{k' \neq k} \left(u_{k'} + v_{k'} c_{k'\uparrow}^{+} c_{-k'\downarrow}^{+}\right) |0\rangle = \boxed{0}$$

in a gauge with $u_k, v_k \in \mathbb{R}$.

$ab + ba = 1$

Charge of quasiparticles

$$\hat{Q} = \sum_{k\sigma} e\, c^+_{k\sigma} c_{k\sigma}$$

$$[\hat{Q}, \gamma^+_{k\uparrow}] = \sum_{p\sigma} e [c^+_{p\sigma} c_{p\sigma}, u_k c^+_{k\uparrow} - v_k^* c_{-k\downarrow}] =$$

$$= \sum_{p\sigma} e \Big(c^+_{p\sigma} \underbrace{c_{p\sigma} u_k c^+_{k\uparrow}}_{\delta_{pk}\delta_{\sigma\uparrow} + c^+_{k\uparrow} c_{p\sigma}} - c^+_{p\sigma} c_{p\sigma} v_k^* c_{-k\downarrow} -$$

$$u_k c^+_{k\uparrow} c^+_{p\sigma} c_{p\sigma} + v_k^* \underbrace{c_{-k\downarrow} c^+_{p\sigma}}_{\delta_{p,-k}\delta_{\sigma\downarrow} - c^+_{p\sigma} c_{-k\downarrow}} c_{p\sigma} \Big) =$$

$$= \sum_{p\sigma} e \Big(c^+_{p\sigma} \delta_{pk} \delta_{\sigma\uparrow} u_k + \cancel{u_k c^+_{p\sigma} c^+_{k\uparrow} c_{p\sigma}} - \cancel{v_k^* c^+_{p\sigma} c_{p\sigma} c_{-k\downarrow}} +$$

$$+ \cancel{u_k c^+_{k\uparrow} c^+_{p\sigma} c_{p\sigma}} + v_k^* c_{p\sigma} \delta_{p,-k} \delta_{\sigma\downarrow} - \cancel{v_k^* c^+_{p\sigma} c_{-k\downarrow} c_{p\sigma}} \Big)$$

(−) (−)

$$= e(u_k c^+_{p\uparrow} + v_k^* c_{-k\downarrow}) \neq e\, \gamma^+_{k\uparrow} \quad !$$

$\gamma^+_{k\sigma} |\Psi_{BCS}\rangle$ – is not an eigenstate of a well defined charge.

Spin of quasiparticles

$$[\hat{S}^z, \gamma^+_{k\uparrow}] = [\tfrac{\hbar}{2} \sum_{\sigma} \sigma\, c^+_{p\sigma} c_{p\sigma}, \hat{\gamma}^+_{k\uparrow}] = \tfrac{\hbar}{2} \hat{\gamma}^+_{k\uparrow}$$

$\hat{\gamma}^+_{k\sigma} |\Psi_{BCS}\rangle$ has a well defined spin state.

Cooper pair

$S^z = 0, \quad Q = 2e$! spin-charge separation.

c) <u>Gap equation</u>

$$\Delta_k = -\frac{1}{N}\sum_{k'} V_{kk'} \langle c_{-k'\downarrow} c_{k'\uparrow} \rangle =$$

$$= -\frac{1}{N}\sum_{k'} V_{kk'} \Big\langle \left(-v_{k'} \gamma^+_{k'\uparrow} + u_{k'} \gamma_{-k'\downarrow} \right) \left(u_{k'} \gamma_{k'\uparrow} + v_{k'} \gamma^+_{-k'\downarrow} \right) \Big\rangle =$$

$$= -\frac{1}{N}\sum_{k'} V_{kk'} \Big\{ -v_{k'} u_{k'} \langle \gamma^+_{k'\uparrow} \gamma_{k'\uparrow} \rangle - v_{k'}^2 \langle \gamma^+_{k'\uparrow} \gamma^+_{-k'\downarrow} \rangle +$$

$$+ u_{k'}^2 \langle \gamma_{-k'\downarrow} \gamma_{k'\uparrow} \rangle + u_{k'} v_{k'} \langle \gamma_{-k'\downarrow} \gamma^+_{-k'\downarrow} \rangle \Big\}$$

But $\langle \gamma^+_{k\uparrow} \gamma_{k\uparrow} \rangle = f(E_k) = \dfrac{1}{e^{\beta E_k}+1}$

in equilibrium $\langle \gamma^+_{k\uparrow} \gamma^+_{-k'\downarrow} \rangle = 0$

$\langle \rangle = \frac{1}{Z} \mathrm{Tr}[e^{\beta H} \ldots]$ $\quad \langle \gamma_{-k'\downarrow} \gamma_{k'\uparrow} \rangle = 0$

$$\langle \gamma_{k'\downarrow} \gamma^+_{-k'\downarrow} \rangle = 1 - f(E_k)$$

$$\boxed{\Delta_k = -\frac{1}{N}\sum_{k'} V_{kk'} u_{k'} v_{k'} [1 - 2 f(E_{k'})] = -\frac{1}{N}\sum_{k'} V_{kk'} \frac{\Delta_{k'}}{2E_{k'}} [1 - 2 f(E_{k'})]}$$

Solution for a BCS model

$$\boxed{1 = V_0 \int_{-\omega_D}^{\omega_D} d\xi \, \rho(\xi+\mu) \, \frac{1 - 2f\left(\sqrt{\xi^2+\Delta_0^2}\right)}{2\sqrt{\xi^2+\Delta_0^2}} =}$$

$$= V_0 \int_{-\omega_D}^{\omega_D} d\xi \, \rho(\mu+\xi) \, \frac{\tanh\left(\frac{\beta}{2}\sqrt{\xi^2+\Delta_0^2}\right)}{2\sqrt{\xi^2+\Delta_0^2}} \simeq$$

$$\simeq V_0 \, \rho(\varepsilon_F) \int_{-\omega_D}^{\omega_D} d\xi \, \frac{\tanh \frac{\beta}{2}\sqrt{\xi^2+\Delta_0^2}}{2\sqrt{\xi^2+\Delta_0^2}}$$

For $T=0$ ($\beta=\infty$)

$$\boxed{\Delta_0(0) = 2\,\omega_D \, e^{-\frac{1}{V_0\rho(\varepsilon_F)}}}$$

At $T=T_c$ ($\Delta_0 = 0$)

$$1 = V_0 \, \rho(\varepsilon_F) \int_{-\omega_D}^{\omega_D} d\xi \, \frac{\tanh\left(\frac{\xi}{2k_BT_c}\right)}{2\xi} =$$

$$= V_0 \, \rho(\varepsilon_F) \int_0^{\beta\omega_D/2} dx \, \frac{\tanh x}{x} = \text{by parts}$$

$$= V_0\,\rho(\varepsilon_F)\left\{ \ln\frac{\beta_c\omega_D}{2} \underbrace{\tanh\frac{\beta\omega_D}{2}}_{\simeq 1} - \underbrace{\int_0^{\beta\omega_D/2 \to \infty} dx \, \frac{\ln x}{\cosh^2 x}}_{\ln\frac{\pi}{4} - \gamma} \right\}$$

$$\Rightarrow \boxed{k_BT_c \simeq \frac{2e^{\gamma}}{\pi}\,\omega_D \, e^{-\frac{1}{V_0\rho(\varepsilon_F)}}}$$

the same as from binding energy of Cooper pair!

$$\boxed{\frac{2\Delta_0(0)}{k_BT_c} \simeq \frac{2\pi}{e^{\gamma}} \simeq 3.528}$$

$$\boxed{\frac{\Delta_0(0)}{k_BT_c} \simeq 1.76}$$

	$T_c(K)$	$\hbar\omega_D/k_BT$	$\rho(\varepsilon_F)V_0$	Δ/k_BT_c
Cd	0.56	164	0.18	1.6
Al	1.2	375	0.18	1.3 - 2.1
Sn	3.75	185	0.25	1.6
Pb	7.22	96	0.38	2.2

§8 Predictions of BCS model

•) isotop effect

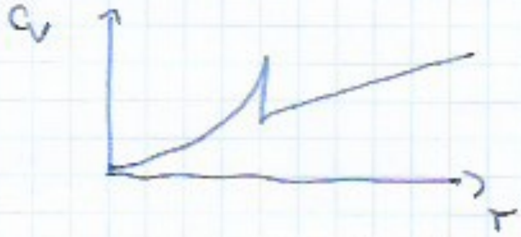
$$T_c,\ \Delta_0 \sim \omega_D e^{-\frac{1}{\rho V_0}}$$

for single oscillator $\omega \sim \frac{1}{\sqrt{m}}$

for a lattice $\omega_{q\lambda} \sim \frac{1}{\sqrt{M}} \rightarrow \omega_D \sim \frac{1}{\sqrt{M}}$

$$\rightarrow \boxed{T_c \sim \frac{1}{\sqrt{M}}}$$

•) specific heat

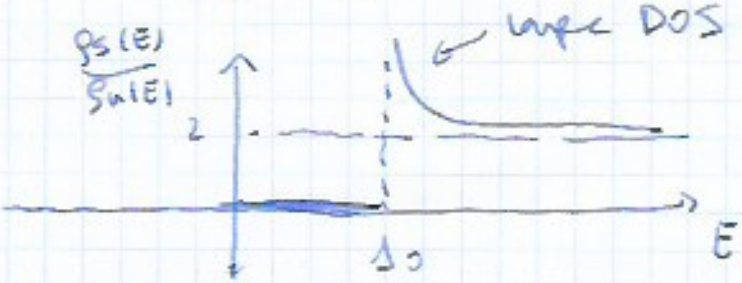


$$\boxed{\Delta C = \frac{8\pi^2}{7\zeta(3)} \rho(\epsilon_F) k_B^2 T_c}$$

•) tunnelling density of states

$$\rho_s(E) = \frac{1}{N}\sum_k \delta(E - E_k) = \frac{1}{N}\sum_k \delta\left(E - \sqrt{\xi_k^2 + |\Delta_k|^2}\right) =$$

$$= \begin{cases} \rho_n(\epsilon_F)\dfrac{2E}{\sqrt{E^2 - \Delta_0^2}} & E > \Delta_0 \\ 0 & E < \Delta_0 \end{cases}$$



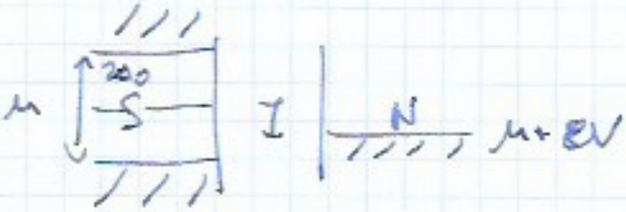
- tunneling current

N | I | N

$I_{NN} = G_{NN} V$
↑ conductance

T = 0

$$G_{NN} = \frac{2e^2}{h} \rho_L^n(\xi_F)\rho_R^n(\xi_F)|t|^2$$



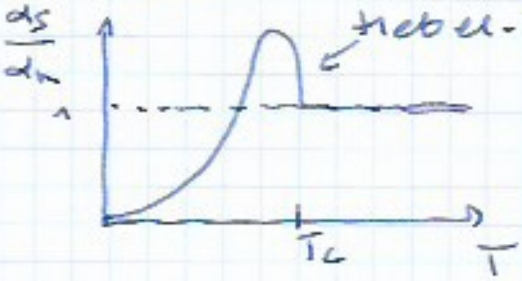
$$G_{SN} = G_{NN} \frac{\rho_L^s(|eV|)}{\rho_R^n(\xi_F)}$$ ← follows a DOS

T = 0

- ultrasonic attenuation (absorption)

$$\frac{\alpha_s}{\alpha_n} = 2f(\Delta_0) = \frac{2}{e^{\beta\Delta_0}+1}$$

- nuclear relaxation - relaxation of nuclear spin NMR



Hebel-Slichter peak due to large DOS