Gauge symmetry and coherence of $|\Psi_{BCS}\rangle$

A key role in SC is played by the average

$$\langle c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow} \rangle$$

If this average is taken on a state with N-particles $|FS\rangle_N = |0\rangle$

$${}_N\langle FS| c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow} |FS\rangle_N \sim {}_N\langle FS|FS\rangle_{N+2} = 0$$

Different subspaces in $\mathcal{F} = \bigoplus_{N=0}^{\infty} \mathcal{H}_N$ are orthogonal.

However, for $|\Psi_{BCS}\rangle = \prod_k (u_k + v_k c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow})|0\rangle$

$$\langle \Psi_{BCS}| c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow} |\Psi_{BCS}\rangle = u_k v_k^* \neq 0$$

<u>why?</u>

Let's write up to a global (irrelevant) phase with <u>$u_k, v_k \in \mathbb{R}$</u>

$$|\Psi^\phi_{BCS}\rangle = \prod_k (u_k + v_k e^{i\phi} c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow})|0\rangle =$$

$$= \mathcal{N} \prod_k \left(1 + e^{i\phi} \frac{v_k}{u_k} c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow}\right)|0\rangle =$$

$\boxed{\text{Pauli principle} \quad (c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow})^{n>1} = 0}$

$$= \mathcal{N} \prod_k \exp\left(e^{i\phi} \frac{v_k}{u_k} c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow}\right)|0\rangle =$$

$$= \mathcal{N} \exp\Big(e^{i\phi} \underbrace{\sum_k \frac{v_k}{u_k} c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow}}_{=:\, b^\dagger}\Big)|0\rangle = \mathcal{N} e^{e^{i\phi} b^\dagger}|0\rangle$$

(*) Remember a coherent state in harmonic oscillator

$$\boxed{|\alpha\rangle} = e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n (e^{i\phi})^n}{n!} (\hat{a}^{\dagger})^n |0\rangle =$$

$$= e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} (e^{i\phi})^n |n\rangle =$$

$\hat{H}|n\rangle = E_n|n\rangle$

$E_n = \hbar\omega_0(n+1/2)$

$$\boxed{= e^{-\frac{|\alpha|^2}{2}} e^{e^{i\phi}\alpha \hat{a}^{\dagger}} |0\rangle}$$

Eigenstate of the annihilation operator

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$$

and

$$\langle\alpha|\hat{a}|\alpha\rangle = \alpha$$

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$\boxed{|\Psi^{\phi}_{BCS}\rangle \text{ – a generalized coherent state of Cooper pairs}}$

However, <u>Wigner superselection rule</u> forbids a superposition of particles (states) with different masses / charges.

Yes, but for finite systems.

In $|\Psi^{\phi}_{BCS}\rangle$ we are working in <u>grand canonical ensemble</u> and in <u>thermodynamic limit.</u>

Nevertheless, we can project $|\Psi^{\phi}_{BCS}\rangle$ onto a state with a fixed N (P. W. Anderson)

$$|\Psi_N\rangle = \int_0^{2\pi} d\phi \, e^{-iN\phi} \, |\Psi_\phi\rangle$$

Computing earlier.

$$b^+ = \sum_k \frac{v_k}{u_k} c^+_{k\uparrow} c^+_{-k\downarrow} = \qquad c_{k\sigma} = \frac{1}{\sqrt{V}} \int dx \, e^{ikx} \psi(x)$$

$$= \int dx_1 \int dx_2 \, \psi^+_\uparrow(x_1) \psi^+_\downarrow(x_2) \underbrace{\frac{1}{V} \sum_k \frac{v_k}{u_k} e^{-ik(x_1-x_2)}}_{=: \varphi(x_1-x_2)} =$$

$$= \int dx_1 \int dx_2 \, \varphi(x_1-x_2) \psi^+_\uparrow(x_1) \psi^+_\downarrow(x_2)$$

wave function of a single Cooper pair

$$|\Psi_N\rangle = \int_0^{2\pi} d\phi \, e^{-iN\phi} \mathcal{N} e^{e^{i\phi} b^+} |0\rangle = \qquad \frac{1}{2\pi} \int_0^{2\pi} d\phi \, e^{-i\phi(N-M)} = \delta_{NM}$$

$$= \mathcal{N} \sum_{M=0}^{\infty} \int_0^{2\pi} d\phi \, e^{-iN\phi} \frac{e^{i\phi M} (b^+)^M}{M!} |0\rangle =$$

$$= \mathcal{N} \frac{2\pi}{N!} (b^+)^N |0\rangle =$$

$$= 2\pi \frac{\mathcal{N}}{N!} \int dx_1 \ldots \int dx_{2N} \, \underline{\varphi(x_1-x_2)\varphi(x_3-x_4) \ldots \varphi(x_{2N-1}-x_{2N})} \psi^+_\uparrow(x_1)\psi^+_\downarrow(x_2) \ldots \psi^+_\uparrow(x_{2N-1})\psi^+_\downarrow(x_{2N})|0\rangle$$

We find N Cooper pairs in the same state φ like bosons in BEC condensate occupy $k=0$ state $\rightarrow$ <u>macroscopic coherence</u>

In $|\Psi_{BCS}\rangle$ the electromagnetic gauge invariance is spontaneously broken (violated) (???)

(*) Remember, $[\underbrace{\hat{H}}_{\text{Hamiltonian}}, \underbrace{\hat{O}}_{\text{symmetry operation}}] = 0$

$$\hat{H}|G\rangle = E_g|G\rangle \qquad |G\rangle \text{ - ground state}$$

$$\hat{O}\hat{H}|G\rangle = E_g\hat{O}|G\rangle$$

$$\hat{H}\underbrace{\hat{O}|G\rangle}_{|G\rangle} = E_g\underbrace{\hat{O}|G\rangle}_{|G\rangle} \qquad \text{the same ground state}$$

In systems with broken symmetries

$$\hat{O}|G\rangle \neq |G\rangle \quad \text{though } [\hat{H}, \hat{O}] = 0$$

e.g. Ferromagnet - $\hat{O} = \hat{O}(3)$ - rotations

$$|\uparrow\uparrow\uparrow\cdots\uparrow\rangle \neq |\downarrow\downarrow\downarrow\cdots\downarrow\rangle$$

Gauge symmetry in electrodynamics

$$\begin{cases} \bar{\nabla}\cdot\bar{E} = \rho/\varepsilon_0 \\ \bar{\nabla}\times\bar{E} + \frac{\partial\bar{B}}{\partial t} = 0 \\ \bar{\nabla}\cdot\bar{B} = 0 \\ \bar{\nabla}\times\bar{B} - \mu_0\varepsilon_0\frac{\partial\bar{E}}{\partial t} = \mu_0\bar{j} \end{cases} \qquad \begin{aligned} \bar{B} &= \bar{\nabla}\times\bar{A} \\ \bar{E} &= -\frac{\partial\bar{A}}{\partial t} - \bar{\nabla}V \end{aligned}$$

$$\boxed{\begin{aligned} \bar{A}' &= \bar{A} + \bar{\nabla}\chi \\ V' &= V - \frac{\partial\chi}{\partial t} \end{aligned}}$$

Gauge symmetry in quantum mechanics

$$\hat{H} = \frac{1}{2m}\sum_\sigma\int dx\, \psi_\sigma^\dagger(x)\left(-i\hbar\bar{\nabla} - i\frac{e}{\hbar}\bar{A}(x)\right)^2\psi_\sigma(x) + \hat{H}_{int.}$$

$$|\Psi\rangle = \sum_{\sigma_1\ldots\sigma_N}\int dx_1\ldots dx_N\, \varphi(x_1\ldots x_N, \sigma_1\ldots\sigma_N)\,\psi^\dagger_{\sigma_1}(x_1)\ldots\psi^\dagger_{\sigma_N}(x_N)|0\rangle$$

$$\boxed{\varphi'(x_1\ldots x_N, \sigma_1\ldots\sigma_N) \Rightarrow \prod_{j=1}^{N} e^{-i\frac{e}{\hbar}\chi}\,\varphi(x_1\ldots x_N, \sigma_1\ldots\sigma_N)}$$

The field operators transform as

$$\psi_\sigma^\dagger(x) \to e^{i\frac{e}{\hbar}\chi(x)}\,\psi_\sigma^\dagger(x)$$

$$\psi_\sigma(x) \to e^{-i\frac{e}{\hbar}\chi(x)}\,\psi_\sigma(x)$$

<u>In case of $|\psi_{BEC}^{\phi}\rangle$ it means</u>

$$c_{k\sigma}^\dagger \to e^{i\frac{e}{\hbar}\chi}\,c_{k\sigma}^\dagger$$

$$\phi \to \phi - \frac{2e}{\hbar}\chi$$

Then $|\psi_{BEC}^{\phi}\rangle = \prod_k \left(u_k + v_k e^{i\phi} c_{k\uparrow}^\dagger c_{-k\downarrow}^\dagger\right)|0\rangle$

is gauge invariant.

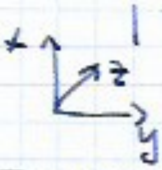
It is merely the label ϕ which is adjusted under a gauge transformation!

Rotation of coordinate axis	$\Longleftrightarrow$	Gauge transformation

$|\uparrow\uparrow\uparrow\dots\uparrow\rangle$

$\updownarrow$ the same ground state

$|\uparrow\uparrow\uparrow\dots\uparrow\rangle$



transformation of our description of a system from one gauge to another, without ever having any effect on the physical state of the system.

M. Greiter, Ann. Phys. 319, 217 (2005)