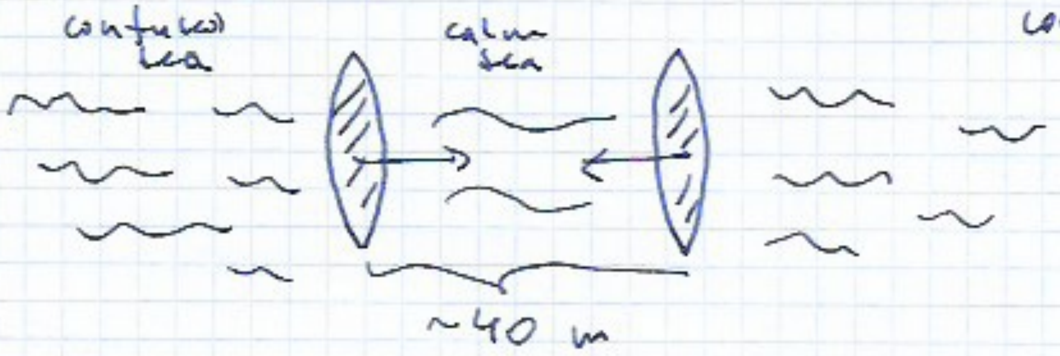
# CASIMIR FORCE

fluctuation-induced forces: Casimir force,
van der Waals force
Casimir-Polder force

→ XVIII cat. observation - two ships, no wind
confused sea

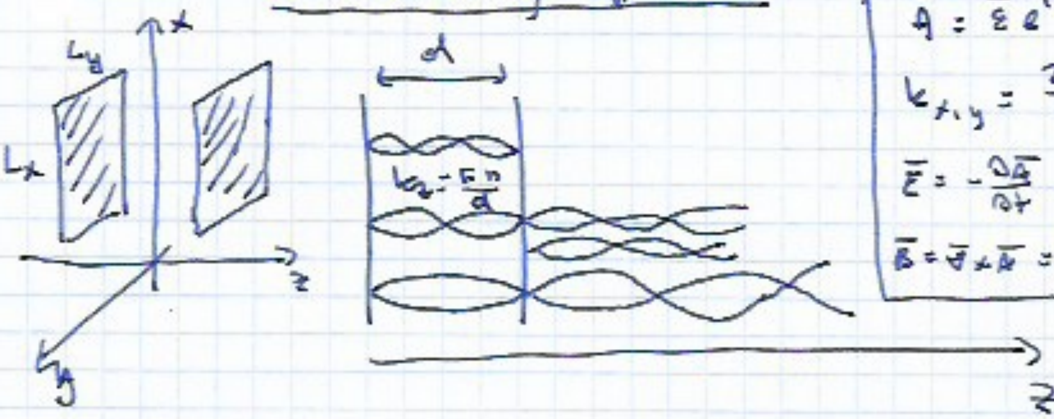



plastic island has the same origin of plastic islands

→ Fluctuating charges in neutral body give rise to fluctuating electromagnetic fields which interacts with charges of other bodies. -
dispersion force

e.g. Gecko on most slippery surfaces -
milions of tiny keratin hairs on each foot
collectively produces adhesive effect
also adhesive tapes.

## Casimir force between two perfectly conducting plates




$$\boxed{\begin{aligned}
&\bar{A} = \bar{\varepsilon}\, e^{ik_x x} e^{ik_y y} \sin(k_z z)\, e^{-i\omega_{\bar{k}} t} \\
&k_{x,y} = \frac{2\pi}{L_{x,y}} n_{x,y}, \quad k_z = \frac{\pi}{d} n_z \\
&\bar{E} = -\frac{\partial \bar{A}}{\partial t} = i\omega_{\bar{k}} \bar{\varepsilon}\, e^{ik_x x} e^{ik_y y} \sin(k_z z)\, e^{-i\omega_{\bar{k}} t} \\
&\bar{B} = \bar{\nabla} \times \bar{A} = \bar{k} \times \bar{\varepsilon}\, e^{ik_x x} e^{ik_y y} \sin(k_z z)\, e^{-i\omega_{\bar{k}} t} \\
&\omega_{\bar{k}} = c|\bar{k}|
\end{aligned}}$$

due to the boundary condition $\bar{E}_{\parallel}\big|_{z} = 0$, $\bar{B}_{\perp}\big|_{z} = 0$

$$k_z = \frac{n\pi}{d}, \quad n \in \mathbb{Z} \text{ is quantized}$$

### How to calculate the force?

Assume that this force is potential, so

$$F_z(d) = -\left.\frac{dU}{dz}\right|_{z=d}$$

We need to get the potential energy of the plate interaction $U(d)$, which shows how much energy we need to use to separate two plates from ~~dz~~ $z = d$ to $z \to \infty$

$$\boxed{U(d) = E(d) - E(d \to \infty)}$$

For quantized electromagnetic field we know the Hamiltonian

$$\hat{H} = \sum_{\vec{k}s} \hbar \omega_{\vec{k}} \left[ \hat{a}^{+}_{\vec{k}s} \hat{a}_{\vec{k}s} + \frac{1}{2} \right]$$

[ zero mode energy

$\omega_{\vec{k}} = c|\vec{k}|$, $s$ - polarization

$$\left[ \hat{a}_{\vec{k}s}, \hat{a}^{+}_{\vec{k}'s'} \right] = \delta_{\vec{k}\vec{k}'} \delta_{ss'}$$

$$E = \langle 0 | \hat{H} | 0 \rangle$$ - energy of the system in its ground state $|0\rangle$ (no real photons)

For QED we have

$$\omega_{\vec{k}} = c|\vec{k}| = c\sqrt{k_x^2 + k_y^2 + k_z^2} = c\sqrt{k_x^2 + k_y^2 + \left(\frac{n\pi}{a}\right)^2}$$

Therefore,

$$E = \sum_{\vec{k}s} \frac{\hbar \omega_{\vec{k}}}{2} = \hbar c A \int_{-\infty}^{\infty} \frac{dk_x}{2\pi} \int_{-\infty}^{\infty} \frac{dk_y}{2\pi} \sum_{n=0}^{\infty} \sqrt{k_x^2 + k_y^2 + \left(\frac{n\pi}{d}\right)^2}$$

since

$$\sum_s \to 2 \qquad \sum_{\vec{k}} = \sum_{k_x k_y} = \frac{L_x}{2\pi}\int_{-\infty}^{\infty} dk_x \frac{L_y}{2\pi}\int_{-\infty}^{\infty} dk_y, \quad A = L_x L_y$$

$$\sum_{k_z} f(k_z) = \sum_{n=0}^{\infty} f\left(\frac{\pi n}{d}\right)$$ ← countable infinity

In cylindrical coordinates $\int dk_x \int dk_y = \int d\varphi \int dk_\rho k_\rho = 2\pi \int dk_\rho k_\rho$

$$\frac{E}{A} = \frac{E_s(d)}{A}\bigg|_{s=0} = \frac{\hbar c}{2\pi} \sum_{n=0}^{\infty} \int_0^{\infty} dk_\rho k_\rho \left( \sqrt{k_\rho^2 + \left(\frac{\pi n}{d}\right)} \right)^{1-s} \Bigg|_{s=0}$$

$s$ - a parameter, would be complex

$$\int \sum_n^{\infty} \leftrightarrow \sum_n^{\infty} \int$$ if everything converges (?!)

If $\mathrm{Re}\, s > 3$ then the result is finite but in the end we need $s \to 0$.

Let's try this trick, we introduce a formal parameter, perform an analytic continuation on complex plane ($s \in \mathbb{C}$) such that sum and integral are finite, calculate $u(d)$ and set $s \to 0$ $\longleftrightarrow$ renormalization

$$\frac{E_s(d)}{A} = \left\{ \begin{array}{l} x = k_\rho^2 \\ dx = 2k_\rho\, dk_\rho \end{array} \right\} =$$

$$= \frac{\hbar c}{2\pi} \sum_{n=0}^{\infty} \frac{1}{2} \int_0^{\infty} dx \left[ x + \left(\frac{\pi n}{d}\right)^2 \right]^{\frac{1-s}{2}} =$$

$$= \frac{\hbar c}{4\pi} \sum_{n=0}^{\infty} \frac{2}{3-s} \left[ x + \left(\frac{\pi n}{d}\right)^2 \right]^{\frac{3-s}{2}} \Bigg|_0^{\infty} =$$

$$= -\frac{\hbar c}{2\pi} \frac{1}{3-s} \Bigg[ \sum_{n=0}^{\infty} \left(\frac{\pi n}{d}\right)^{3-s} - \underbrace{\sum_{n=0}^{\infty} \lim_{k_\rho \to \infty} \underbrace{\left[k_\rho^2 + \left(\frac{\pi n}{d}\right)^2\right]^{\frac{3-s}{2}}}_{\sim k_\rho^{3-s}}}_{\text{diverges for } s \to 0} \Bigg]$$

We need $u(d) = E(d) - E(d \to \infty)$

$$E(d) \underset{d \to \infty}{=} \dots \sqrt{k_x^2 + k_y^2 + \lim_{d \to \infty} \left(\frac{\pi n}{d}\right)^2} = \dots \sqrt{k_\rho^2 + 0}$$

this term reduces to the second divergent term

Therefore,

$$\frac{U(d)}{A} = -\lim_{s\to 0} \frac{\hbar c}{2\pi} \frac{1}{3-s} \frac{\pi^{3-s}}{d^{3-s}} \sum_{n=0}^{\infty} n^{3-s} =$$

$$= -\frac{\hbar c \pi^2}{6 d^3} \sum_{n=0}^{\infty} n^3 = -\frac{\hbar c \pi^2}{720 d^3}$$

We used regularized Riemann zeta-function

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} \quad \text{- converges if } \operatorname{Re} s > 1$$

$$\zeta(-3) = \frac{1}{120} \leftarrow \text{from analytical continuation}$$

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$$\zeta(s) = \sum_{k=1}^{\infty} k^{-s} = \qquad \operatorname{Re} s > 1$$

$$= -\frac{\Gamma(1-s)}{2\pi i} \int_C \frac{(-z)^{s-1}}{e^z - 1} dz =$$

$$= 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s) \zeta(1-s) =$$

$$= \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{x^{s-1}}{e^x - 1} dx \; ; \; \operatorname{Re} s > 1$$

$$= \frac{1}{(1-2^{1-s})\Gamma(s)} \int_0^{\infty} \frac{x^{s-1}}{e^x + 1} dx$$

e.g.

$$\zeta(-1) = 2^{-1} \pi^{-2} (-1) \underbrace{\Gamma(2)}_{1} \underbrace{\zeta(2)}_{\pi^2/6} =$$

$$= -\frac{1}{12}$$

$$\zeta(-3) = 2^{-3} \pi^{-4} (1) \underbrace{\Gamma(4)}_{6} \underbrace{\zeta(4)}_{\pi^4/90} =$$

$$= \frac{1}{120}$$

$\zeta(0) = -\frac{1}{2}$

$\zeta(1) = \infty$

$\zeta(-2n) = 0$

$\zeta(1-2n) = -\frac{B_{2n}}{2n}$

$\zeta(2n) = \frac{(2\pi)^{2n}}{2(2n)!} |B_{2n}|$

$\zeta(2) = \pi^2/6$

$\zeta(4) = \pi^4/90$

Bernulli numbers

$$\frac{t e^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}$$

$$B_n = B_n(0)$$

$$B_0 = 1, \; B_1 = -\frac{1}{2}, \; B_4 = -\frac{1}{30}$$

# Analytical continuation

Df. two analytic function defined on $D_1$ and $D_2$

$$f_1 : D_1 \to V_1$$

$$f_2 : D_2 \to V_2$$


If there is $U = D_1 \cap D_2 \neq \emptyset$ such that

$$\forall z \in U \quad f_1(z) = f_2(z)$$

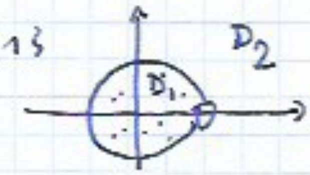
then we say that $f_2$ is analytic continuation of $f_1$ and vice versa.

Example: Geometric series

$$f_1(z) = 1 + z + z^2 + \dots = \sum_{n=0}^{\infty} z^n \quad \text{converges if}$$

$$D_1 = \{ z \in \mathbb{C} \mid |z| < 1 \}$$

$$f_2(z) = \frac{1}{1-z} \quad \text{defined on } D_2 = \mathbb{C} \setminus \{1\}$$




On $U = D_1 \cap D_2 \quad f_1(z) = f_2(z)$.

$f_2$ is analytic continuation of $f_1$ on to $D_2$.

Using $f_2$ one can get results for divergent series

$$1 - 1 + 1 - 1 + \dots = f_1(-1) = f_2(-1) = \frac{1}{1-(-1)} = \frac{1}{2}$$

$$1 + 2 + 4 + 8 + \dots = f_1(2) = f_2(2) = \frac{1}{1-2} = -1$$

An interpretation is needed!

Force is attractive and per unit area (pressure)

$$\boxed{\frac{F_z(d)}{A} = \frac{\hbar c \pi^2}{240 d^4}}$$

$A = 1\,\mu m^2$

$d = 5\,nm$

$F \sim 0.1\ mN$ extremely strong in nano world!

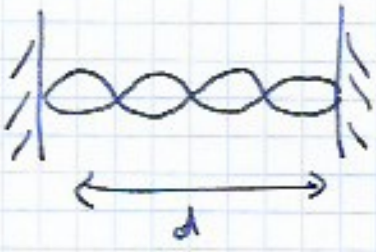
If plates are finite, not perfect conducting we might get

$F \sim 1\ nN \sim 10^{-9} N$ strong enough to break a molecule

With atomic force microscopy we can measure $F \sim pN \sim 10^{-12}\ N$

## Toy model in $d=1$

$$\hat{H} = \sum_{n\sigma} \hbar\omega_n \left(\hat{a}^\dagger_n \hat{a}_n + \tfrac{1}{2}\right)$$




$$n\,\frac{\lambda}{2} = L$$

$$\omega_n = c\,\frac{n\pi}{d}$$

$$E(d) = \langle 0|\hat{H}|0\rangle = \langle 0|\, \hbar c \frac{\pi}{d} \sum_{n s} n \left(\underbrace{\hat{a}^\dagger_n \hat{a}_n}_{0} + \tfrac{1}{2}\right)|0\rangle =$$

$\langle 0|0\rangle$

$\sum_s = 2$

$$= \frac{\hbar c \pi}{d} \sum_{n=0}^{\infty} n$$

$$E(d\to\infty) = \frac{\hbar c \pi}{d} \int_0^\infty dn\, n$$

$$U(d) = E(d) - E(d\to\infty) = \frac{\hbar c\pi}{d}\Big(\underbrace{\sum_{n=0}^{\infty} n}_{\infty} - \underbrace{\int_0^\infty n\,dn}_{\infty}\Big) = ?$$

How to make sense for

$$\Delta(f) = \sum_{n=0}^{\infty} f(n) - \int_0^\infty f(n)\,dn \;?$$

### Regularization of $\sum_{n=0}^{\infty} f(n)$

$\to$ heat kernel regularization $\quad \tilde{f}(s) = \sum_{n=0}^{\infty} f(n)\underbrace{e^{-ns}}_{\text{cutoff function}}$

$$\sum_{n=0}^{\infty} n e^{-ns} = -\frac{d}{ds}\sum_{n=0}^{\infty}(e^{-s})^n = -\frac{d}{ds}\frac{1}{1-e^{-s}} = \frac{e^s}{(e^s-1)^2} \xrightarrow[s\to 0]{} \frac{1}{s^2} - \frac{1}{12} + O(s^2)$$

$$\int_0^\infty n e^{-sn}\,dn = -\frac{d}{ds}\int_0^\infty e^{-sn}\,dn = -\frac{d}{ds}\,\frac{1}{-s}\,e^{-sn}\Big|_0^\infty = \frac{1}{s^2}$$

$$\left[U(d) = \frac{\hbar c\pi}{d}\left(\frac{1}{s^2} - \frac{1}{12} + O(s^2) - \frac{1}{s^2}\right) \xrightarrow[s\to 0]{} -\frac{\hbar c\pi}{12 d}\right]$$

$$\boxed{F_z = \frac{dU}{dd} = \frac{\hbar c \pi}{12 d^2}}$$

At high ω a metal is no longer perfect conductor

ii) $\zeta$ - regularization

$$\tilde{f}(s) = \sum_{n=0}^{\infty} f(n) \underbrace{n^{-s}}_{\text{cutoff function}} \qquad \text{with } \operatorname{Re} s > 2$$

$$\sum_{n=0}^{\infty} n\, n^{-s} = \zeta(s-1) \xrightarrow[\text{analytic continuation}]{} \zeta(-1) = -\frac{1}{12}$$

$$\int_0^{\infty} n\, n^{-s}\, dn = \int_0^{\infty} n^{1-s}\, dn = \left.\frac{n^{2-s}}{2-s}\right|_0^{\infty} = 0 \quad \text{for } \operatorname{Re} s > 2$$

Regularization should not be dependent on the used method!

iii) Euler - Maclaurin formula

$$\boxed{\sum_{n=0}^{\infty} f(n) - \int_0^{\infty} f(n)\, dn = -\sum_{k=1}^{\infty} \frac{B_k}{k!} f^{(k-1)}(0)}$$

$B_k$ - Bernoulli numbers

$B_0 = 1$ $\quad B_2 = \frac{1}{6}$

$B_1 = -\frac{1}{2}$

$f(n) = n$ then

$$\sum_{n=0}^{\infty} n - \int_0^{\infty} f(n)\, dn = -\frac{B_1}{1!} n\Big|_{n=0} - \frac{B_2}{2!} = -\frac{1}{12}$$

With a suitable cutoff function one can justify applying E-M formula to divergent series

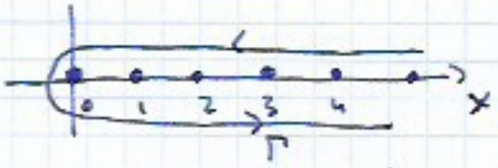
## •v) Abel - Plana formula

Cauchy integral formula

$$f(\zeta)=\frac{1}{2\pi i}\oint_{\Gamma_\zeta}\frac{f(z)}{z-\zeta}\,dz \qquad \text{Res}$$

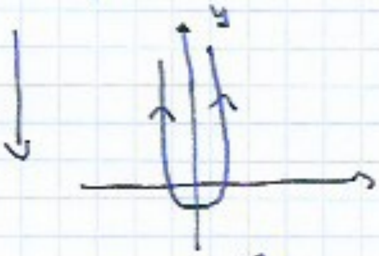
with $\quad \pi\cot(\pi z)=\sum_{n=-\infty}^{\infty}\frac{1}{z-n}$

we get $\quad \sum_{n=0}^{\infty} f(n)=\frac{1}{2\pi i}\sum_{n=0}^{\infty}\oint_{\Gamma_n}\frac{f(z)}{z-n}\,dz=\frac{1}{2i}\int_{\Gamma}\cot(\pi z)f(z)\,dz$





rotation $\pm\pi/2$




$$\sum_{n=0}^{\infty} f(n)=\frac{1}{2}f(0)+\frac{i}{2}\int_0^{\infty}\big(f(iy)-f(-iy)\big)\cot(\pi y)\,dy$$

Similarly $\qquad \cot(ix)=\frac{e^{ix}+e^{-ix}}{e^{ix}-e^{-ix}}$

$$\int_0^{\infty} f(n)\,dn=\frac{i}{2}\int_0^{\infty}\big(f(iy)-f(iy)\big)\,dy$$

Hence

$$\boxed{\sum_{n=0}^{\infty} f(n)-\int_0^{\infty} f(n)\,dn=\frac{1}{2}f(0)+i\int_0^{\infty}\frac{f(iy)-f(-iy)}{e^{2\pi y}-1}\,dy}$$

$f(n)=n$ then $\text{Res}=-2\underbrace{\int_0^{\infty}\frac{y\,dy}{e^{2\pi y}-1}}_{\frac{1}{24}}=-\frac{1}{12}$

to justify one needs a cutoff function

$$\sum_{n=0}^{\infty} n-\int_0^{\infty} n\,dn=-\frac{1}{12}$$

P