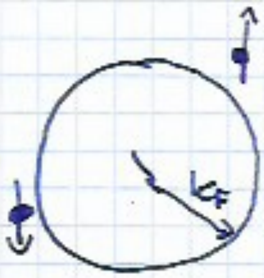
Cooper pair

A symplified model showing formation of a bound state - a Cooper pair



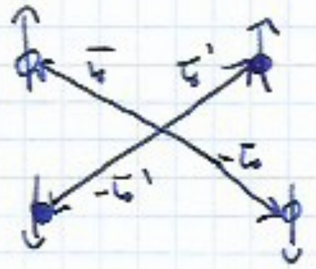
Variational (trial) wave function/state

$$|\Psi\rangle = \sum_{\vec{k} > k_F} \alpha_{\vec{k}} \, c^{+}_{-\vec{k}\downarrow} c^{+}_{\vec{k}\uparrow} | FS \rangle$$

Model Hamiltonian with interaction

$$\hat{H} = \sum_{\vec{k}\sigma} \varepsilon_{\vec{k}} c^{+}_{\vec{k}\sigma} c_{\vec{k}\sigma} + \sum_{\vec{k}\vec{k}'} V_{\vec{k}\vec{k}'} c^{+}_{\vec{k}\uparrow} c^{+}_{-\vec{k}\downarrow} c_{-\vec{k}'\downarrow} c_{\vec{k}'\uparrow}$$

$V_{\vec{k}\vec{k}'}$ - scattering potential, if $V_{\vec{k}\vec{k}'} < 0$ attractive



Normalization

$c_{k\sigma} c^\dagger_{k'\sigma'} + c^\dagger_{k'\sigma'} c_{k\sigma} = \delta_{kk'}\delta_{\sigma\sigma'}$

$$1 = \langle\Psi|\Psi\rangle = \sum_{kk'}{}' \alpha_k^* \alpha_{k'} \langle FS| c_{k\uparrow} \underbrace{c_{-k\downarrow} c^\dagger_{-k'\downarrow}}_{\delta_{kk'} - c^\dagger_{-k'\downarrow} c_{-k\downarrow}} c^\dagger_{k'\uparrow} |FS\rangle =$$

$$= \sum_{kk'}{}' \alpha_k^* \alpha_{k'} \langle FS| \underbrace{c_{k\uparrow} c^\dagger_{k'\uparrow}}_{\delta_{kk'} - c^\dagger_{k'\uparrow} c_{k\uparrow}} \delta_{kk'} - c_{k\uparrow} c^\dagger_{-k'\downarrow} \underbrace{c_{-k\downarrow} c^\dagger_{k'\uparrow}}_{(-)} |FS\rangle =$$

$$= \sum_{kk'}{}' \alpha_k^* \alpha_{k'} \langle FS| \delta_{kk'} - \overbrace{c^\dagger_{k'\uparrow} c_{k\uparrow} + c_{k\uparrow} c^\dagger_{-k'\downarrow} c^\dagger_{k'\uparrow} \underbrace{c_{-k\downarrow}}_{=0}}^{=0} |FS\rangle =$$

$$= \sum_{kk'}{}' \delta_{kk'} \alpha_k^* \alpha_{k'} \langle FS|FS\rangle = \boxed{\sum_k |\alpha_k|^2 = 1}$$

Matrix elements

$$\boxed{\sum_{k\sigma} \varepsilon_k c^\dagger_{k\sigma} c_{k\sigma} \sum_q{}' \alpha_q c^\dagger_{-q\downarrow} c^\dagger_{q\uparrow} |FS\rangle} =$$

$$= \sum_{k\sigma} \sum_q{}' \varepsilon_k \alpha_q \, c^\dagger_{k\sigma} \left(\delta_{k,-q} \delta_{\sigma\downarrow} - c^\dagger_{-q\downarrow} c_{k\sigma}\right) c^\dagger_{q\uparrow} |FS\rangle =$$

$$= \sum_{k\sigma} \sum_q{}' \varepsilon_k \alpha_q \left(c^\dagger_{k\sigma} c^\dagger_{q\uparrow} \delta_{k,-q} \delta_{\sigma\downarrow} - c^\dagger_{k\sigma} c^\dagger_{-q\downarrow} \underbrace{c_{k\sigma} c^\dagger_{q\uparrow}}\right) |FS\rangle =$$

$$= \sum_{k\sigma} \sum_q{}' \varepsilon_k \alpha_q \delta_{k,-q} \delta_{\sigma\downarrow} c^\dagger_{k\sigma} c^\dagger_{q\uparrow} |FS\rangle -$$

$$- \sum_{k\sigma} \sum_q{}' \varepsilon_k \alpha_q c^\dagger_{k\sigma} c^\dagger_{-q\downarrow} \left(\delta_{kq} \delta_{\sigma\uparrow} - \underbrace{c^\dagger_{q\uparrow} c_{k\sigma}}_{=0}\right) |FS\rangle =$$

$$= \sum_q{}' \underbrace{\varepsilon_{-q}}_{=\varepsilon_q} \alpha_q c^\dagger_{-q\downarrow} c^\dagger_{q\uparrow} |FS\rangle - \sum_q{}' \varepsilon_q \alpha_q \underbrace{c^\dagger_{q\uparrow} c^\dagger_{-q\downarrow}}_{(-)} |FS\rangle =$$

$$= \boxed{2 \sum_q{}' \varepsilon_q \alpha_q c^\dagger_{-q\downarrow} c^\dagger_{q\uparrow} |FS\rangle}$$

$\varepsilon_k = \varepsilon_{-k}$ Kramers symmetry

time-reversal symmetry

$$\sum_{kk'} V_{kk'}\, c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow} c_{-k'\downarrow} c_{k'\uparrow} \sum_l{}' \alpha_l\, c^\dagger_{-l\downarrow} c^\dagger_{l\uparrow} |FS\rangle =$$

$$= -\sum_{kk'}\sum_l{}' V_{kk'}\alpha_l\, c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow} c_{-k'\downarrow} c^\dagger_{-l\downarrow}\left(\delta_{k'l} - c^\dagger_{l\uparrow} \underbrace{c_{k'\uparrow})|FS\rangle}_{\to 0}\right.$$

$$= -\sum_{kk'}\sum_l{}' V_{kk'}\alpha_l\, c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow}\, \delta_{k'l}\left(\delta_{-k',-l} - c^\dagger_{-l\downarrow} \underbrace{c_{-k'\downarrow})|FS\rangle}_{\to 0}\right.$$

$$= -\sum_k\sum_l{}' V_{kl}\alpha_l\, c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow}|FS\rangle = \boxed{+\sum_{kl}\alpha_l V_{kl}\, c^\dagger_{-k\downarrow} c^\dagger_{k\uparrow}|FS\rangle}$$

only $k=l$

$\Rightarrow$

$$\langle\Psi|\hat{H}|\Psi\rangle = \sum_{kl}{}' 2\varepsilon_l \alpha_l \alpha_k^* \langle FS|\, c_{k\uparrow} c_{-k\downarrow} c^\dagger_{-l\downarrow} c^\dagger_{l\uparrow}|FS\rangle$$

$$+\sum_{kk'l}\alpha_l \alpha_{k'}^* V_{kl} \langle FS|\, c_{k'\uparrow} c_{-k'\downarrow} c^\dagger_{-k\downarrow} c^\dagger_{k\uparrow}|FS\rangle =$$

only $k=k'$

energy functional

$$= \sum_k 2\varepsilon_k|\alpha_k|^2 + \sum_{kk'} V_{kk'}\alpha_k^*\alpha_{k'} \equiv E(\{\alpha_k\})$$

$$\frac{\partial}{\partial\alpha_k^*}\left[E(\{\alpha_k\}) - \lambda\left(\sum_k|\alpha_k|^2-1\right)\right] = 0$$

variational principle

$\uparrow$ Lagrange multiplayer

$$\boxed{2\varepsilon_k\alpha_k + \sum_{k'} V_{kk'}\alpha_{k'} - \lambda\alpha_k = 0}$$

$$\boxed{\alpha_{\vec k} = - \frac{\sum_{k'} V_{\vec k \vec k'}\, \alpha_{\vec k'}}{2\varepsilon_{\vec k} - \lambda}}$$

integral equation on $\alpha_{\vec k}$ with eigenvalue λ

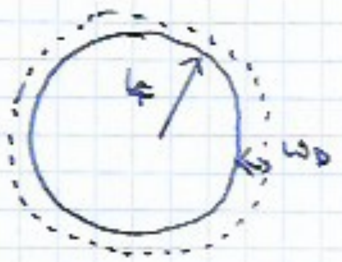
Model potential

attraction

Debye cutoff

$$V_{\vec k \vec k'} = \begin{cases} -V_0 & \varepsilon_k < \hbar\omega_D \\ 0 & \text{otherwise} \end{cases}$$

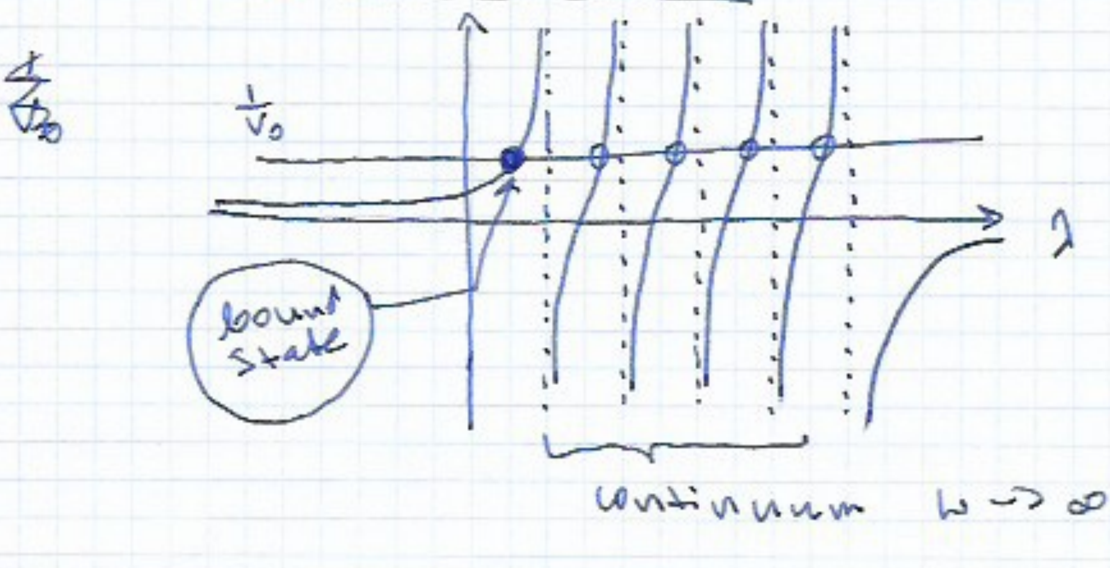
$V_0 > 0$



$$\alpha_k = V_0 \frac{\sum_{k'} \alpha_{k'}}{2\varepsilon_k - \lambda} \qquad / \sum_k$$

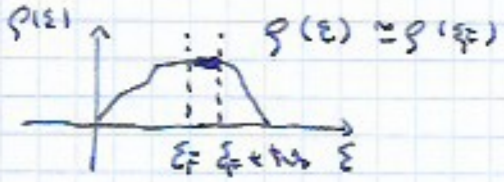
$$\underbrace{\sum_k \alpha_k} = V_0 \sum_k \frac{1}{2\varepsilon_k - \lambda} \underbrace{\sum_{k'} \alpha_{k'}}$$

$$\boxed{\frac{1}{V_0} = \sum_k \frac{1}{2\varepsilon_k - \lambda}}$$



Solution

$$\frac{1}{V_0} = \int_{\varepsilon_F}^{\varepsilon_F+\hbar\omega_D} d\varepsilon \, \frac{\overbrace{\rho(\varepsilon)}^{\text{DOS}}}{2\varepsilon-\lambda} \simeq \rho(\varepsilon_F)\int_{\varepsilon_F}^{\varepsilon_F+\hbar\omega_D} d\varepsilon \, \frac{1}{2\varepsilon-\lambda} =$$



$$= \rho(\varepsilon_F)\,\frac{1}{2}\ln\left|\frac{2(\varepsilon_F+\hbar\omega)-\lambda}{2\varepsilon_F-\lambda}\right|$$

$$\lambda = 2\varepsilon_F - 2\hbar\omega_D\,\frac{e^{-\frac{2}{\rho(\varepsilon_F)V_0}}}{1-e^{-2/\rho(\varepsilon_F)V_0}} \underset{V_0\rho(\varepsilon_F)\ll 1}{\simeq}$$

$$\simeq 2\varepsilon_F - 2\hbar\omega_D\, e^{-\frac{2}{V_0\rho(\varepsilon_F)}}$$

What is λ ?

$$E(\{\alpha_k\}) - \lambda\sum_k |\alpha_k|^2 + \lambda =$$

$$= \sum_k 2\varepsilon_k|\alpha_k|^2 + \sum_{kk'} V_{kk'}\alpha_k^*\alpha_{k'} - \lambda\sum_k|\alpha_k|^2 + \lambda =$$

$$= \sum_k \alpha_k^*\big(\underbrace{2\varepsilon_k\alpha_k + \sum_{k'} V_{kk'}\alpha_{k'} - \lambda\alpha_k}_{=0}\big) + \lambda = \lambda$$

λ is the energy at the minimum

$$\boxed{E = 2\varepsilon_F - 2\hbar\omega_D\, e^{-\frac{2}{V_0\rho(\varepsilon_F)}}}$$

two electrons at the Fermi surface

binding energy

(*) non-perturbative result, $\rho V_0 \ll 1$ not expand above

Schrödinger equation

$$\hat{H}\,|\Psi\rangle = E\,|\Psi\rangle$$

$$\Big[\sum_{k\sigma} \varepsilon_k c_{k\sigma}^{+} c_{k\sigma} + \sum_{kk'} V_{kk'} c_{k\uparrow}^{+} c_{-k\downarrow}^{+} c_{-k'\downarrow} c_{k'\uparrow}\Big] \sum_{l} \alpha_l c_{-l\downarrow}^{+} c_{l\uparrow}^{+} |FS\rangle =$$

$$= E \sum_{l} \alpha_l c_{-l\downarrow}^{+} c_{l\uparrow}^{+} |FS\rangle$$

$$\sum_{l}{}' 2\varepsilon_l \alpha_l c_{-l\downarrow}^{+} c_{l\uparrow}^{+} |FS\rangle + \sum_{kl}{}' \alpha_l V_{kl} c_{-k\downarrow}^{+} c_{k\uparrow}^{+} |FS\rangle =$$

$$= \sum_{l} E \alpha_l c_{-l\downarrow}^{+} c_{-l\uparrow}^{+} |FS\rangle$$

$$\sum_{l}{}' \underbrace{\Big[2\varepsilon_l \alpha_l + \sum_{k} V_{lk} \alpha_k - E\alpha_l\Big]}_{=0} c_{-l\downarrow}^{+} c_{l\uparrow}^{+} |FS\rangle = 0$$

$$\boxed{\alpha_l = -\frac{\sum_{k} V_{lk} \alpha_k}{2\varepsilon_l - E}}$$

the same integral equation as before

The trial wave function $|\Psi\rangle$ is apparently an <u>exact</u> state of $\hat{H}$.