§5. Exactly solvable limits of the Hubbard model in arbitrary dimensions

$$\hat{H} = \sum_{ij\sigma} t_{ij}\, \hat{a}^{+}_{i\sigma} \hat{a}_{j\sigma} + U \sum_{i} \hat{n}_{i\uparrow} \hat{n}_{i\downarrow}$$

$$t_{ij} = \langle w_i | \frac{\hat{\vec{p}}^2}{2m} + V_1 | w_j \rangle =$$

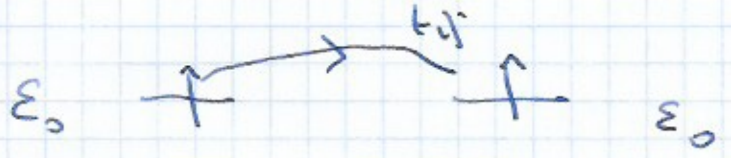
$$= \int dr\, w_i^*(\vec{r}) \left[-\frac{\hbar^2}{2m} \nabla^2 + V_1(\vec{r}) \right] w_j(\vec{r})$$

If $\underline{i = j}$ $\qquad t_{ii} = \int dr\, w_i^*(\vec{r}) \left[\frac{\hat{\vec{p}}^2}{2m} + V_1 \right] w_i(\vec{r}) \equiv \varepsilon_0$

ε_0 atomic energy

$i \neq j \quad t_{ij} = \int dr\, w_i^*(\vec{r}) \left[\frac{\hat{\vec{p}}^2}{2m} + V_1 \right] w_j(\vec{r})$

hopping amplitude

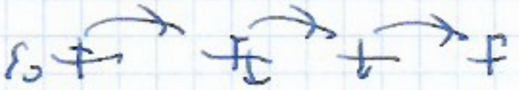


$$U = \int dr \int dr'\, w_i^*(\vec{r}) w_i^*(\vec{r}\,') V_2(|\vec{r} - \vec{r}\,'|) w_i(\vec{r}\,') w_i(\vec{r})$$

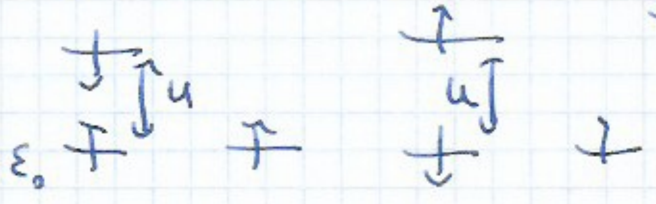
local (on site) interaction energy

The model is solvable in $U = 0$ and $t_{ij} = 0$

free electrons $\qquad$ atomic limit



metallic/covalent bonds



separated atoms

Free electrons $U=0$

$$\hat{H} = \varepsilon_0 \sum_{i\sigma} \hat{n}_{i\sigma} + \sum_{ij\sigma} t_{ij} \hat{a}^+_{i\sigma} \hat{a}_{j\sigma}$$

To diagonalize the Hamiltonian we (discrete) Fourier transform it

$$\hat{a}_{i\sigma} = \frac{1}{\sqrt{N_L}} \sum_{\bar{k}} e^{i\bar{k}\cdot\bar{R}_i} \hat{a}_{\bar{k}\sigma}$$

$$\hat{a}^+_{i\sigma} = \frac{1}{\sqrt{N_L}} \sum_{\bar{k}} e^{-i\bar{k}\cdot\bar{R}_i} \hat{a}^+_{\bar{k}\sigma}$$

$$\sum_{i\sigma} n_{i\sigma} = \frac{1}{N_L} \sum_{i\sigma} \sum_{\bar{k}\bar{k}'} e^{-i\bar{k}\cdot\bar{R}_i} e^{i\bar{k}'\bar{R}_i} \hat{a}^+_{\bar{k}\sigma} \hat{a}_{\bar{k}'\sigma} =$$

$$= \sum_{\bar{k}\bar{k}'\sigma} \hat{a}^+_{\bar{k}\sigma} \hat{a}_{\bar{k}'\sigma} \underbrace{\frac{1}{N_L} \sum_i e^{-i(\bar{k}-\bar{k}')\bar{R}_i}}_{\delta_{kk'}} = \sum_{\bar{k}\sigma} \hat{n}_{\bar{k}\sigma}$$

$$\sum_{ij\sigma} t_{ij} \hat{a}^+_{i\sigma} \hat{a}_{j\sigma} = \sum_{\bar{k}\bar{k}'\sigma} \hat{a}^+_{\bar{k}\sigma} \hat{a}_{\bar{k}'\sigma} \frac{1}{N} \sum_{ij} t_{ij} e^{-i\bar{k}\bar{R}_i} e^{i\bar{k}'\bar{R}_j} =$$

$$= \sum_{\bar{k}\bar{k}'\sigma} \hat{a}^+_{\bar{k}\sigma} \hat{a}_{\bar{k}'\sigma} \frac{1}{N} \sum_{ij} t_{ij} e^{-i\bar{k}(\bar{R}_i-\bar{R}_j)} e^{-i(\bar{k}-\bar{k}')\bar{R}_j} =$$

$$= \sum_{\bar{k}\bar{k}'\sigma} \hat{a}^+_{\bar{k}\sigma} \hat{a}_{\bar{k}'\sigma} \underbrace{\sum_{i(j)} t_{ij} e^{-i\bar{k}(\bar{R}_i-\bar{R}_j)}}_{\varepsilon_{\bar{k}}} \underbrace{\frac{1}{N} \sum_j e^{-i(\bar{k}-\bar{k}')\bar{R}_j}}_{\delta_{\bar{k}\bar{k}'}} =$$

$$t_{ij} = t(|\bar{R}_i - \bar{R}_j|)$$

$$= \sum_{\bar{k}\sigma} \varepsilon_{\bar{k}} \hat{n}_{\bar{k}\sigma}$$

$$\boxed{\varepsilon_{\bar{k}} = \sum_{j(i)} t_{ij}\, e^{i\bar{k}(\bar{R}_i-\bar{R}_j)}}$$ dispersion relation

Tight-binding models:

t_{ij}

$$t_{ij} = \begin{cases} -t & i,j \in \langle i,j \rangle \quad \text{n.n.} \\ -t' & i,j \in \langle\langle i,j \rangle\rangle \quad \text{n.n.n.} \\ \text{etc.} & \end{cases}$$

$t, t' > 0$

n.n. models

$$\varepsilon_{\vec{k}} = t\Big(e^{ik_xa} + e^{-ik_xa} + e^{ik_ya} + e^{-ik_ya} + \vdots \Big)$$

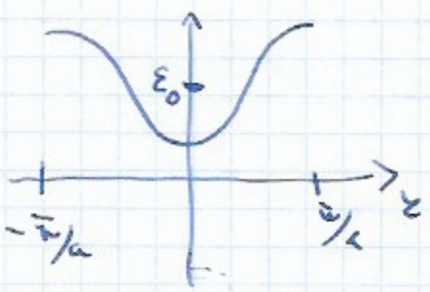
$$\varepsilon_{\vec{k}} = -2t\left(\cos(k_xa) + \cos(k_ya) + \cos(k_za) \dots\right) =$$

$$= -2t \sum_{i=1}^{d} \cos(k_i a)$$

n.n.n. model in $d=2$

$$\varepsilon_{\vec{k}} = -2t\left[\cos(k_xa) + \cos(k_ya)\right] - 4t' \cos\left(\frac{k_xa}{2}\right)\cos\left(\frac{k_ya}{2}\right)$$

$$\hat{H} = \sum_{\vec{k}\sigma} \left(\varepsilon_{\vec{k}} + \varepsilon_0\right) \hat{a}^{\dagger}_{\vec{k}\sigma} \hat{a}_{\vec{k}\sigma}$$



usually $\varepsilon_0 = 0$
sets the energy
level

Thermodynamics – grand canonical ensemble

partition function

$$\Xi(T,\mu,N_L) = \mathrm{Tr}\, e^{-\beta(\hat{H}-\mu\hat{N})} =$$

Fock space

$$\{\otimes |n_{\vec{k}\sigma}\rangle\} = \mathcal{F} \qquad = \prod_{\vec{k}\sigma} \langle n_{\vec{k}\sigma}| e^{-\beta(\varepsilon_{\vec{k}}-\mu)\hat{n}_{\vec{k}\sigma}} |n_{\vec{k}\sigma}\rangle =$$

$$n_{\vec{k}\sigma} = 0,1 \qquad = \prod_{\vec{k}\sigma}\left(1 + e^{-\beta(\varepsilon_k-\mu)}\right)$$

grand canonical potential

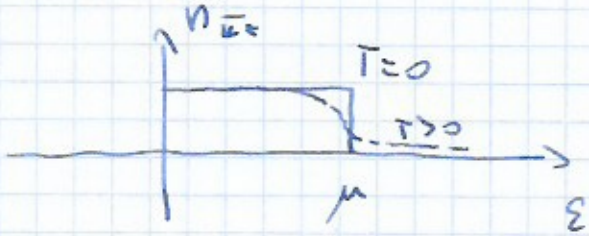
$$\Omega(T,\mu,N_L) = -k_BT \ln \Xi(T,\mu,N_L) =$$

$$= -k_BT \sum_{\vec{k}\sigma} \ln\left(1 + e^{-\beta(\varepsilon_{\vec{k}}-\mu)}\right)$$

$$N = -\left(\frac{\partial\Omega}{\partial\mu}\right)_{T,N_L} = +\cancel{k_BT}\sum_{\vec{k}\sigma}\frac{\cancel{\beta}\, e^{-\beta(\varepsilon_k-\mu)}}{1+e^{-\beta(\varepsilon_{\vec{k}}-\mu)}} =$$

$$= \sum_{\vec{k}\sigma}\frac{1}{e^{\beta(\varepsilon_{\vec{k}}-\mu)}+1} = \sum_{\vec{k}\sigma}\overline{n_{\vec{k}\sigma}}$$

F-D distribution



band filling $\qquad n = \frac{N}{N_L} \in \langle 0, 2\rangle$

$n=0$ empty band $\quad \varepsilon_F < -W/2$

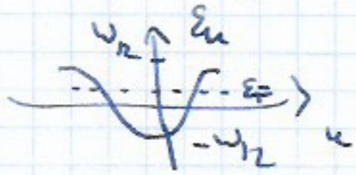
$n=2$ full band $\quad \varepsilon_F > W/2$

} insulators

$n=1$ half filled band $\quad -\frac{W}{2} < \varepsilon_F < W/2$

n other

} metals



Green function at $U=0$

$$G_{ij\sigma}(\tau) = -\langle T\hat{a}_{i\sigma}(\tau)\,\hat{a}^\dagger_{j\sigma}(0)\rangle$$

$$\hat{a}_{i\sigma}(\tau) = e^{\tau\hat{H}}\,\hat{a}_{i\sigma}e^{-\tau\hat{H}}$$

equation of motion

$$\frac{\partial\hat{a}_{i\sigma}(\tau)}{\partial\tau} = [\hat{H},\hat{a}_{i\sigma}(\tau)]$$

$$\hat{H} = \sum_{ij\sigma} t_{ij}\,\hat{a}^\dagger_{i\sigma}\hat{a}_{j\sigma} = \sum_{\vec{k}\sigma}\varepsilon_{\vec{k}}\,\hat{n}_{\vec{k}\sigma}$$

We need FT

$$G_{ij\sigma}(\tau) = -\frac{1}{N}\sum_{kk'}\langle T\,\hat{a}_{k\sigma}(\tau)\,\hat{a}^\dagger_{k'\sigma}(0)\rangle e^{i\vec{k}\vec{R}_i}e^{-i\vec{k}'\vec{R}_j}$$

$$\partial_\tau\hat{a}_{k\sigma}(\tau) = \sum_{\vec{k}'\sigma'}\varepsilon_{\vec{k}'}[\hat{n}_{\vec{k}'\sigma'},a_{\vec{k}\sigma}] = -\varepsilon_{\vec{k}}\,\hat{a}_{\vec{k}\sigma}(\tau)$$

$$\Rightarrow\quad\hat{a}_{\vec{k}\sigma}(\tau) = e^{-\varepsilon_{\vec{k}}\tau}\,a_{\vec{k}\sigma}$$

$$T\,\hat{a}_{k\sigma}(\tau)\,\hat{a}^\dagger_{k'\sigma}(0) = \begin{cases}\hat{a}_{\vec{k}\sigma}(\tau)\,\hat{a}^\dagger_{k'\sigma}(0) & \tau>0\\ -\hat{a}^\dagger_{k'\sigma}(0)\,\hat{a}_{k\sigma}(\tau) & \tau<0\end{cases}$$

$$G_{ij\sigma}(\tau>0) = -\frac{1}{N}\sum_{kk'}e^{-\varepsilon_k\tau}\underbrace{\langle\hat{a}_{k\sigma}\hat{a}^\dagger_{k'\sigma}\rangle}_{(1-f_{\vec{k}})\delta_{kk'}}e^{i\vec{k}\vec{R}_i-i\vec{k}'\vec{R}_j} =$$

$$= \frac{1}{N}\sum_{\vec{k}}e^{i\vec{k}\cdot(\vec{R}_i-\vec{R}_j)}\Big(\underbrace{(-1)(1-f_{\vec{k}})e^{-\varepsilon_{\vec{k}}\tau}}_{G_{\vec{k}\sigma}(\tau>0)}\Big)$$

$$G_{ij\sigma}(\tau<0) = \frac{1}{N}\sum_{\bar k} e^{i\bar k(\bar\ell_i - \bar\ell_j)} \underbrace{f_{\bar k} e^{-\varepsilon_{\bar k}\tau}}_{G_{\bar k}(\tau<0)}$$

for fermions $\quad G(\tau+\beta) = -G(\tau)$ antiperiodic

$$G(\tau) = \frac{1}{\beta}\sum_{i\omega_n} G(i\omega_n) e^{-i\omega_n\tau}$$

$$G(i\omega_n) = \int_0^\beta \frac{d\tau}{\cancel{\beta}} G(\tau) e^{i\omega_n\tau}$$

$$\rightarrow \quad G_{ij}(\tau) = \frac{1}{N}\sum_k e^{i\bar k\cdot(\bar\ell_i - \bar\ell_j)} \quad G_{\bar k\sigma}(\tau)$$

$$G_{\bar k\sigma}(\tau) = \theta(\tau)\left(-e^{-\varepsilon_k\tau}(1-f_k)\right) - \theta(-\tau)\left(e^{-\varepsilon_k\tau} f_{\bar k}\right)$$

$$G_{\bar k\sigma}(i\omega_n) = -\cancel{\beta}\int_0^\beta d\tau\,(1-f_{\bar k}) e^{(i\omega_n - \varepsilon_{\bar k})\tau} =$$

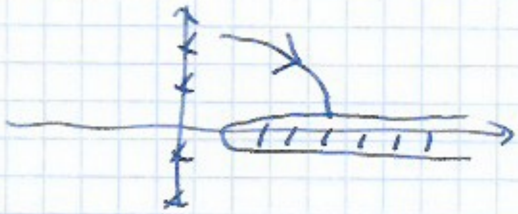
$$= -\cancel{\beta}\left(1 - \frac{1}{e^{\beta\varepsilon_k}+1}\right)\frac{e^{(i\omega_n-\varepsilon_k)\beta} - 1}{i\omega_n - \varepsilon_{\bar k}} = \qquad e^{i\omega_n\beta} = e^{(2n+1)i\frac{\pi}{\beta}\beta} = -1$$

$$= \cancel{\beta}\,\frac{e^{\beta\varepsilon_{\bar k}}}{e^{\beta\varepsilon_k}+1}\;\frac{e^{-\varepsilon_{\bar k}\beta}+1}{i\omega_n - \varepsilon_{\bar k}} =$$

$$= \cancel{\beta}\,\frac{1}{\cancel{e^{-\beta\varepsilon_k}+1}}\;\frac{\cancel{e^{-\beta\varepsilon_{\bar k}}+1}}{i\omega_n - \varepsilon_{\bar k}} = \cancel{\beta}\,\frac{1}{i\omega_n - \varepsilon_{\bar k}}$$

$$\boxed{G_{k\sigma}(i\omega_n) = \frac{1}{i\omega_n - \varepsilon_{\bar k}}}$$

$i\omega_n = \omega + i0^+$ analytic continuation

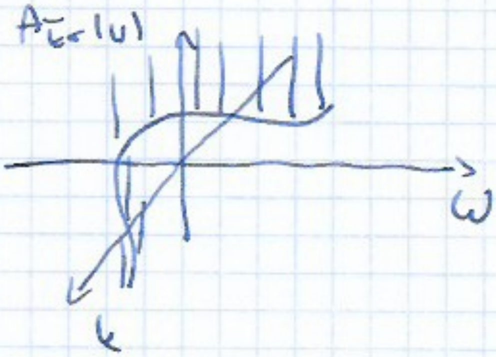


$$\boxed{G^R_{\vec{k}\sigma}(\omega) = \frac{1}{\omega - \varepsilon_{\vec{k}} + i0^+}}$$

<u>Spectral function</u>

$$\boxed{A_{\vec{k}\sigma}(\omega) = -\frac{1}{\pi} \operatorname{Im} G_{\vec{k}\sigma}(\omega)}$$

$$A_{\vec{k}\sigma}(\omega) = \frac{1}{\pi} \frac{(0^+)}{(\omega - \varepsilon_{\vec{k}})^2 + (0^+)^2} = \delta(\omega - \varepsilon_{\vec{k}})$$



well defined quasiparticles with dispersion $\varepsilon_{\vec{k}}$

<u>Density of states</u> $\rho_\sigma(\omega) = \frac{1}{N_L}\sum_{\vec{k}} \delta(\omega - \varepsilon_{\vec{k}})$

per particle the same as in one body QM

in many-body QM

$$\boxed{\rho_\sigma(\omega) = \frac{1}{N_L}\sum_{\vec{k}} A_{\vec{k}\sigma}(\omega)}$$

Atomic limit $t_{ij} = 0$

$$\hat{H} = \sum_{i\sigma} \varepsilon\, \hat{n}_{i\sigma} + U \sum_i \hat{n}_{i\uparrow} \hat{n}_{i\downarrow} = \sum_i \hat{H}_i$$

$$[\hat{H}, \hat{n}_{i\sigma}] = 0 \to \hat{n}_{i\sigma} \text{ conserved on each site!}$$

Fock space

$$\mathcal{F} = \bigotimes_i \mathcal{F}_i , \quad \mathcal{F}_i = \{ |0\rangle_i, |\uparrow\rangle_i, |\downarrow\rangle_i, |\uparrow\downarrow\rangle_i \}$$

$$\hat{H}_i |0\rangle_i = 0$$

$$\hat{H}_i |\uparrow\rangle_i = \varepsilon$$

$$\hat{H}_i |\downarrow\rangle_i = \varepsilon$$

$$\hat{H}_i |\uparrow\downarrow\rangle_i = 2\varepsilon + U$$

$$\mathcal{Z}(T, \mu, N_L) = \mathrm{Tr}\left[e^{-\beta(\hat{H} - \mu \hat{N})} \right] =$$

$$= \prod_{i=1}^{N_L} \mathrm{Tr}_i \left(e^{-\beta(\hat{H}_i - \mu \hat{n}_i)} \right) =$$

$$= \prod_{i=1}^{N_L} \left(1 + 2 e^{-\beta(\varepsilon - \mu)} + e^{-2\beta(\varepsilon - \mu)} e^{-\beta U} \right)$$

$$\Omega(T, \mu, N_L) = -k_B T \sum_{i=1}^{N_L} \ln \left(1 + 2 e^{-\beta(\varepsilon - \mu)} + e^{-2\beta(\varepsilon - \mu)} e^{-\beta U} \right) =$$

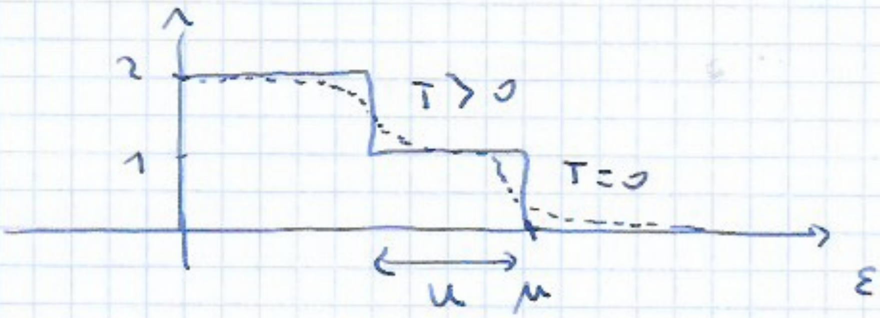
$$= -k_B T N_L \ln \left(1 + 2 e^{-\beta(\varepsilon - \mu)} + e^{-2\beta(\varepsilon - \mu)} e^{-\beta U} \right)$$

$$N = -\left(\frac{\partial \Omega}{\partial \mu} \right)_{T, N_L} = \cancel{k_B T} N_L \frac{2\cancel{\beta} e^{-\beta(\varepsilon - \mu)} + 2\cancel{\beta} e^{-2\beta(\varepsilon - \mu)} e^{-\beta U}}{1 + 2 e^{-\beta(\varepsilon - \mu)} + e^{-2\beta(\varepsilon - \mu)} e^{-\beta U}}$$

$$n(T,\mu) = \frac{N}{N_L} = 2\,\frac{e^{-\beta(\varepsilon-\mu)} + e^{-2\beta(\varepsilon-\mu)}\,e^{-\beta U}}{1 + 2e^{-\beta(\varepsilon-\mu)} + e^{-2\beta(\varepsilon-\mu)}\,e^{-\beta U}}$$

$U = 0$ $$n(T,\mu) = \frac{1}{e^{\beta(\varepsilon-\mu)} + 1}$$

$U = \infty$ $$n(T,\mu) = \frac{1}{\frac{1}{2}e^{\beta(\varepsilon-\mu)} + 1}$$



"Mott plateau"
single occupied

<u>Green function</u>

$$G_{ij\sigma}(\tau) = -\langle T\,\hat{a}_{i\sigma}(\tau)\,\hat{a}^+_{j\sigma}(0)\rangle = +G_{i\sigma}(\tau)\,\delta_{ij}$$

$$\cancel{e^{\tau\hat{H}}\,\hat{a}_{i\sigma}\,e^{-\tau\hat{H}}\,\hat{a}^+_{i\sigma}\,|0\rangle} =$$

$$= \cancel{e^{\tau\hat{H}}\,\hat{a}_{i\sigma}\,e^{-\tau\varepsilon}\,|\sigma\rangle} = \cancel{e^{-\tau\varepsilon}}$$

$$e^{\tau\hat{H}}\,\hat{a}_{i\uparrow}\,e^{-\tau\hat{H}}\,\hat{a}^+_{i\uparrow}\,|\downarrow\rangle =$$

$$= e^{\tau\hat{H}}\,\hat{a}_{i\uparrow}\,e^{-2\tau\varepsilon-\tau U}\,|\uparrow\downarrow\rangle =$$

$$= e^{\tau\hat{H}}\,e^{-2\tau\varepsilon-\tau U}\,|\downarrow\rangle = e^{\tau\varepsilon}\,e^{-2\tau\varepsilon-\tau U}\,|\downarrow\rangle = e^{-\tau\varepsilon-\tau U}\,|\downarrow\rangle$$

Matsubara formalism

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$$\mathcal{H} = \sum_i \left[\varepsilon (n_{i\uparrow} + n_{i\downarrow}) + U n_{i\uparrow} n_{i\downarrow} \right]$$

$$\Xi = \prod_i \left(1 + 2e^{-\beta(\varepsilon-\mu)} + e^{-2\beta(\varepsilon-\mu)} e^{-\beta U} \right) =$$

$$= \left(1 + 2e^{-\beta(\varepsilon-\mu)} + e^{-2\beta(\varepsilon-\mu)} e^{-\beta U} \right)^{N_L}$$

$$\dot{\hat{a}}_{i\sigma} = [\hat{\mathcal{H}}, \hat{a}_{i\sigma}] = -(\varepsilon + U \hat{n}_{i\bar{\sigma}}) \hat{a}_{i\sigma} = \partial_\tau \left[e^{\mathcal{H}\tau} a_{i\sigma} e^{-\mathcal{H}\tau} \right]$$

$$\dot{\hat{n}}_{i\sigma} = [\hat{\mathcal{H}}, \hat{n}_{i\sigma}] = 0$$

$$\rightarrow \quad \hat{a}_{i\sigma}(\tau) = e^{-(\varepsilon + U\hat{n}_{i\bar{\sigma}})} \hat{a}_{i\sigma}$$

$$\langle \hat{n}_{i\sigma} \rangle = \frac{e^{-\beta(\varepsilon-\mu)} + e^{-2\beta(\varepsilon-\mu)} e^{-\beta U}}{1 + 2e^{-\beta(\varepsilon-\mu)} + e^{-2\beta(\varepsilon-\mu)} e^{-\beta U}}$$

$$G_{i\sigma}(\tau) = -\langle T\, a_{i\sigma}(\tau)\, a^{+}_{i\sigma}(0) \rangle =$$

$$= -\frac{1}{\Xi} \mathrm{Tr} \left[e^{-\beta(\hat{\mathcal{H}} - \mu \hat{N})} e^{-(\varepsilon + U\hat{n}_{i\bar{\sigma}})\tau} \hat{a}_{i\sigma} \hat{a}^{+}_{i\sigma} \right] =$$

$$= -\frac{1}{\Xi} \mathrm{Tr} \left[e^{-(\varepsilon - U\hat{n}_{i\bar{\sigma}})\tau} e^{-\beta(\hat{\mathcal{H}} - \mu\hat{N})} (1 - \hat{n}_{i\sigma}) \right] =$$

$$= -\frac{e^{-(\varepsilon-\mu)\tau} + e^{-\beta(\varepsilon-\mu)} e^{-(\varepsilon-\mu+U)\tau}}{1 + 2e^{-\beta(\varepsilon-\mu)} + e^{-2\beta(\varepsilon-\mu)} e^{-\beta U}}$$

$$G_{i\sigma}(i\omega_n) = \int_0^\beta d\tau\, G_{i\sigma}(\tau) e^{i\omega_n\tau} =$$

$$= -\frac{1}{\Xi^{1/N_L}} \left[\frac{-e^{-\beta(\varepsilon-\mu)} - 1}{i\omega_n - (\varepsilon-\mu)} + e^{-\beta(\varepsilon-\mu)} \cdot \frac{-e^{-\beta(\varepsilon-\mu+U)} - 1}{i\omega_n - (\varepsilon-\mu) - U} \right] =$$

$$= \frac{1 - \langle \hat{n}_{i\bar{\sigma}} \rangle}{i\omega_n - (\varepsilon-\mu)} + \frac{\langle \hat{n}_{i\sigma} \rangle}{i\omega_n - (\varepsilon-\mu) - U}$$

Retarded Green's function – equation of motion

$$G_{i\sigma}(t) = -i\,\theta(t)\,\langle\{ a_{i\sigma}(t),\ a^{\dagger}_{i\sigma}\}\rangle =: \langle\langle a_{i\sigma}(t)|\hat{a}^{\dagger}_{i\sigma}\rangle\rangle$$

$$[\hat{a}_{i\sigma}, \hat{H}] = (\varepsilon-\mu)\hat{a}_{i\sigma} + U\,\hat{a}_{i\sigma}\hat{n}_{i\bar{\sigma}} =: \Omega_{+}\hat{a}_{i\sigma}$$

$$[\hat{a}_{i\sigma} n_{i\bar{\sigma}}, \hat{H}] = (\varepsilon-\mu+U)\,\hat{a}_{i\sigma}\hat{n}_{i\bar{\sigma}} =: \Omega_{+}(\hat{a}_{i\sigma}\hat{n}_{i\bar{\sigma}})$$

$$\langle\langle \hat{a}_{i\sigma}|\hat{a}^{\dagger}_{i\sigma}\rangle\rangle_{\omega} = \int_{-\infty}^{\infty} dt\, e^{-i\omega t}\,\langle\langle \hat{a}_{i\sigma}(t)|\hat{a}^{\dagger}_{i\sigma}\rangle\rangle$$

$$\omega\langle\langle \hat{a}_{i\sigma}|\hat{a}^{\dagger}_{i\sigma}\rangle\rangle_{\omega} = 1 + (\varepsilon-\mu)\langle\langle \hat{a}_{i\sigma}|\hat{a}^{\dagger}_{i\sigma}\rangle\rangle_{\omega} + U\langle\langle \hat{a}_{i\sigma}\hat{n}_{i\bar{\sigma}}|\hat{a}^{\dagger}_{i\sigma}\rangle\rangle_{\omega}$$

$$\omega\langle\langle \hat{a}_{i\sigma}\hat{n}_{i\bar{\sigma}}|\hat{a}^{\dagger}_{i\sigma}\rangle\rangle_{\omega} = \langle\hat{n}_{i\bar{\sigma}}\rangle + (\varepsilon-\mu+U)\langle\langle \hat{a}_{i\sigma}\hat{n}_{i\bar{\sigma}}|\hat{a}^{\dagger}_{i\sigma}\rangle\rangle_{\omega}$$

$$\Downarrow$$

$$\langle\langle \hat{a}_{i\sigma}\hat{n}_{i\bar{\sigma}}|\hat{a}^{\dagger}_{i\sigma}\rangle\rangle_{\omega} = \frac{\langle\hat{n}_{i\bar{\sigma}}\rangle}{\omega-\varepsilon+\mu-U}$$

$$G_{i\sigma}(\omega) = \frac{1 + \dfrac{U\langle\hat{n}_{i\bar{\sigma}}\rangle}{\omega-\varepsilon+\mu-U}}{\omega-\varepsilon+\mu} =$$

$$= \frac{1-\langle\hat{n}_{i\bar{\sigma}}\rangle}{\omega-(\varepsilon-\mu)} + \frac{\langle\hat{n}_{i\bar{\sigma}}\rangle}{\omega-(\varepsilon-\mu+U)}$$

$$A_{i\sigma}(\omega) = (1-n_{\bar{\sigma}})\,\delta(\omega-(\varepsilon-\mu)) + n_{i\bar{\sigma}}\,\delta(\omega-(\varepsilon-\mu)-U)$$