The Hubbard model in atomic limit

$$\boxed{\hat{H} = \sum_{\sigma=\uparrow,\downarrow} \varepsilon\, n_\sigma + U\, n_\uparrow n_\downarrow}$$

$\varepsilon + U$ ↑↓

ε ↑ $\quad$ ε ↓ $\quad$ ε —

$$\mathcal{H} = \{ |0\rangle, |\uparrow\rangle, |\downarrow\rangle, |\uparrow\downarrow\rangle \} = \{ |\alpha\rangle \}$$

$$Z = \mathrm{Tr}\, e^{-\beta(\hat{H} - \mu\hat{N})} = \sum_\alpha \langle \alpha | e^{-\beta(\hat{H} - \mu \hat{N})} | \alpha \rangle$$

$$= 1 + e^{-\beta(\varepsilon - \mu)} + e^{-\beta(\varepsilon - \mu)} + e^{-\beta(2\varepsilon + U - 2\mu)}$$

$$= 1 + 2 e^{-\beta(\varepsilon - \mu)} + e^{-\beta(2\varepsilon + U - 2\mu)}$$

$$\boxed{F = -\frac{1}{\beta} \ln Z = -k_B T \ln\left(1 + 2 e^{-\beta(\varepsilon-\mu)} + e^{-\beta(2\varepsilon + U - 2\mu)}\right)}$$

$$\bar{n} = \frac{\mathrm{Tr}\left[\hat{N} e^{-\beta(\hat{H} - \mu\hat{N})}\right]}{\mathrm{Tr}\, e^{-\beta(\hat{H} - \mu\hat{N})}} = \frac{1}{Z}\frac{1}{\beta}\frac{\partial}{\partial\mu} \mathrm{Tr}\, e^{-\beta(\hat{H} - \mu\hat{N})} =$$

$$= \frac{1}{\beta}\frac{\partial}{\partial\mu} \ln Z = -\frac{\partial}{\partial\mu} F =$$

$$\boxed{= \frac{2\left(e^{-\beta(\varepsilon-\mu)} + \left(e^{-\beta(\varepsilon-\mu)}\right)^2 e^{-\beta U}\right)}{1 + 2 e^{-\beta(\varepsilon-\mu)} + \left(e^{-\beta(\varepsilon-\mu)}\right)^2 e^{-\beta U}}}$$

<u>Limits</u>: $U = 0$:

$$F = -2 k_B T \ln\left(1 + e^{-\beta(\varepsilon-\mu)}\right)$$

$$\bar{n} = \frac{2}{e^{\beta(\varepsilon-\mu)} + 1}$$

$U = \infty$: $\quad F = -k_B T \ln\left(1 + 2 e^{-\beta(\varepsilon-\mu)}\right)$

$$\bar{n} = \frac{2}{e^{\beta(\varepsilon-\mu)} + 2}$$

$$\boxed{Z = \int \mathcal{D}[c_\sigma^*, c_\sigma]\, e^{-S[c_\sigma^*, c_\sigma]}}$$

$$\boxed{S[c_\sigma^*, c_\sigma] = \int_0^\beta d\tau \left(\sum_\sigma c_\sigma^* (\partial_\tau - \mu + \varepsilon) c_\sigma + U\, n_\uparrow n_\downarrow \right)}$$

$c_\sigma(\tau+\beta) = -c_\sigma(\tau)$

introduce spinor notation

$$\psi = \begin{pmatrix} c_\uparrow \\ c_\downarrow \end{pmatrix}, \quad \bar{\psi} = (c_\uparrow^*, c_\downarrow^*)$$

$$\bar{\psi}\psi = (c_\uparrow^* \; c_\downarrow^*) \begin{pmatrix} c_\uparrow \\ c_\downarrow \end{pmatrix} = c_\uparrow^* c_\uparrow + c_\downarrow^* c_\downarrow =$$

$$= n_\uparrow + n_\downarrow$$

$$\bar{\psi}\partial_\tau \psi = (c_\uparrow^* \; c_\downarrow^*)\, \partial_\tau \begin{pmatrix} c_\uparrow \\ c_\downarrow \end{pmatrix} =$$

$$= c_\uparrow^* \partial_\tau c_\uparrow + c_\downarrow^* \partial_\tau c_\downarrow$$

$$\frac{U}{2}(\bar{\psi}\psi)^2 = \frac{U}{2}(n_\uparrow + n_\downarrow)^2 = \frac{U}{2}(n_\uparrow^2 + n_\downarrow^2 + 2 n_\uparrow n_\downarrow) =$$

$$= \frac{U}{2}(n_\uparrow + n_\downarrow + 2 n_\uparrow n_\downarrow) =$$

$$= \frac{U}{2}(n_\uparrow + n_\downarrow) + U\, n_\uparrow n_\downarrow$$

$$\Rightarrow U\, n_\uparrow n_\downarrow = \frac{U}{2}(\bar{\psi}\psi)^2 - \frac{U}{2}\bar{\psi}\psi$$

$$\boxed{S = \int_0^\beta d\tau\, \bar{\psi}(\tau)\left(\partial_\tau - \mu - \frac{U}{2} + \varepsilon\right)\psi(\tau) + \frac{U}{2}(\bar{\psi}\psi)^2}$$

$\psi(\tau+\beta) = -\psi(\tau)$

Hubbard Stratonovich transformation

based on the integral

$$\int_{-\infty}^{\infty} dx\, e^{-\frac{a}{2}x^2 + bx} = \sqrt{\frac{2\pi}{a}}\, e^{\frac{b^2}{2a}}$$

Reading backward we can eliminate b^2 term.

$$Z = \int D[\bar\psi,\psi]\, e^{-\int_0^\beta d\tau\, \bar\psi(\partial_\tau - \mu - \frac{U}{2} + \varepsilon)\psi + \frac{U}{2}(\bar\psi\psi)^2}$$

quartic term

$$= \mathcal{N}\int D[\bar\psi\psi]\int D[\phi]\, e^{-\int_0^\beta d\tau\left(\bar\psi(\partial_\tau - \mu - \frac{U}{2} + \varepsilon - iU\phi)\psi + \frac{U}{2}\phi^2\right)}$$

some normalization

$\boxed{\text{PTO}} \longrightarrow$

$$\begin{cases} a \leftrightarrow U \\ b \leftrightarrow iU(\bar\psi\psi) \end{cases}$$

Indeed,

$$\int D[\phi]\, e^{-\int_0^\beta d\tau\left(\frac{U}{2}\phi(\tau)^2 - iU(\bar\psi\psi)\phi(\tau)\right)} =$$

$$= \mathcal{N}^{-1} e^{+\int_0^\beta d\tau \frac{(-iU\bar\psi\psi)^2}{U}} = \mathcal{N}^{-1} e^{-\int_0^\beta d\tau \frac{U}{2}(\bar\psi\psi)^2}$$

$\square$

The last line is quadratic in ψ and can be integrated completely

$$Z = \mathcal{N}\int D[\phi]\left(\mathrm{Det}\left[\partial_\tau - \mu - \frac{U}{2} + \varepsilon - iU\phi(\tau)\right]\right)^2 e^{-\int_0^\beta d\tau \frac{U}{2}\phi(\tau)^2} =$$

$$= \mathcal{N}\int D[\phi]\, e^{-S_{eff}[\phi]}$$

$$\boxed{S_{eff}[\phi] = -2\ln \mathrm{Det}\left[\partial_\tau - \mu - \frac{U}{2} + \varepsilon - iU\phi(\tau)\right] + \frac{U}{2}\int_0^\beta d\tau\, \phi(\tau)^2}$$

Hubbard-Stratonovich - details

$$I = \frac{\int D[\phi]\, e^{-\frac{u}{2}\int_0^\beta d\tau\, \phi(\tau)^2 + \int_0^\beta d\tau\, \phi(\tau)\,\alpha(\tau)}}{\int D[\phi]\, e^{-\frac{u}{2}\int_0^\beta d\tau\, \phi(\tau)^2}}$$

$$\boxed{\begin{array}{c} \alpha = iu\bar{\psi}\psi \\ \phi(\tau)\in\mathbb{R} \\ \phi(\tau+\beta)=\phi(\tau) \end{array}}$$

How to compute this integral?

$$-\frac{u}{2}\phi(\tau)^2 + \alpha(\tau)\phi(\tau) =$$

$$= -\frac{u}{2}\left(\phi(\tau)^2 - 2\frac{\alpha(\tau)}{u}\phi(\tau)\right) =$$

$$= -\frac{u}{2}\left(\phi(\tau)^2 - 2\frac{\alpha(\tau)}{u}\phi(\tau) + \left(\frac{\alpha(\tau)}{u}\right)^2\right) + \frac{\alpha(\tau)^2}{2u}$$

$$I = \frac{\int D[\phi]\, e^{-\frac{u}{2}\int_0^\beta d\tau\left(\phi(\tau) - \frac{\alpha(\tau)}{u}\right)^2}\; e^{\int_0^\beta d\tau \frac{\alpha(\tau)^2}{2u}}}{\int D[\phi]\, e^{-\frac{u}{2}\int_0^\beta d\tau\, \phi(\tau)^2}}$$

Compute

$$\int D[\phi]\, e^{-\frac{u}{2}\int_0^\beta d\tau\, \phi(\tau)^2} = \left\{ \begin{array}{l} \phi(\tau) = \frac{1}{\beta}\sum_n e^{i\omega_n\tau}\phi_n \\ \phi(\tau)^2 = \sum_n \phi_n\phi_{-n} \end{array} \right\} =$$

$\phi_n = a_n + ib_n$, $a_n = a_{-n}$

$\phi_n^* = a_n - ib_n$, $b_n = -b_{-n}$, $b_0 = 0$

$\phi\in\mathbb{R} \Leftrightarrow \phi_{-n}^* = \phi_n$

$$\sum_n \phi_n\phi_{-n} = \sum_n a_n^2 + b_n^2$$

$$= \int_{-\infty}^{\infty} \underbrace{\left|\det\left[\frac{\delta\phi(\tau)}{\delta\phi_n}\right]\right|}_{J} da_0 \prod_{n\geq 1} da_n\, db_n\, e^{-\frac{u}{2\beta}a_0^2 - \sum_{n=1}^{\infty}\frac{u}{\beta}(a_n^2+b_n^2)}$$

$$= J\sqrt{\frac{2\pi\beta}{u}}\prod_{n\geq 1}^{\infty}\left(\frac{\pi\beta}{u}\right)$$

Compute

$$\int D[\phi]\, e^{-\frac{u}{2}\int_0^\beta d\tau \left(\phi(\tau) - \frac{\alpha(\tau)}{u}\right)^2} = \begin{cases} \tilde{\phi}(\tau) = \phi(\tau) - \frac{\alpha(\tau)}{u} \\ \left|\mathrm{Det}\, \frac{d\phi(\tau)}{d\tilde{\phi}(\tau)}\right| = 1 \end{cases}$$

$$= \int D[\tilde{\phi}]\, e^{-\frac{u}{2}\int_0^\beta d\tau\, \tilde{\phi}(\tau)^2} =$$

$$= J \sqrt{\frac{2\pi\beta}{u}} \prod_{n\geq 1}^{\infty} \left(\frac{\pi\beta}{u}\right)$$

Remark: when we do shift $\tilde{\phi} = \phi - \frac{\alpha(\tau)}{u}$

$$\int_{-\infty}^{\infty} da_n\, db_n \longrightarrow \int_{-\infty + \tilde{\alpha}_n}^{+\infty + \tilde{\alpha}_n}$$

$\tilde{\alpha}_n$: Fourier token of

$$\frac{\alpha(\tau)}{2u} = -\frac{iu}{2u}\underbrace{(c_\uparrow^+ c_\uparrow + c_\downarrow^+ c_\downarrow)}_{\text{grassman}}$$

and $\int_{-\infty}^{\infty} e^{-x^2} dx = \int_{-\infty + ia}^{\infty + ia} e^{-x^2} dx = \sqrt{\pi}$

but $c_\sigma^+ c_\sigma = n_\sigma = 0, 1$

$\square$

So, we find that

$$\boxed{e^{\int_0^\beta d\tau \frac{\alpha(\tau)^2}{2u}} = \frac{\int D[\phi]\, e^{-\frac{u}{2}\int_0^\beta d\tau\, \phi(\tau)^2 + \int_0^\beta d\tau\, \alpha(\tau)\phi(\tau)}}{\int D[\phi]\, e^{-\frac{u}{2}\int_0^\beta d\tau\, \phi(\tau)^2}}}$$

$$\alpha(\tau) = iu\bar{\psi}\psi = iu\,(c_\uparrow^+ c_\uparrow + c_\downarrow^+ c_\downarrow)$$

quartic in Grassman

Quadratic in Grassman (Gaussian)

Hubbard-Stratonovich operator identity

$$\boxed{\sqrt{\frac{2\pi}{a}}\; e^{\frac{\hat{b}^2}{2a}} = \int_{-\infty}^{\infty} dx\, e^{-\frac{a}{2}x^2 + \hat{b}x}}$$

quadratic in $\hat{b}$ operator

linear in $\hat{b}$ operator

<u>Proof</u>

$$R = \int_{-\infty}^{\infty} dx\, e^{-\frac{a}{2}x^2 + \hat{b}x} = \sum_{n=0}^{\infty} \frac{\hat{b}^n}{n!} \int_{-\infty}^{\infty} dx\, x^n e^{-\frac{a}{2}x^2} =$$

only even terms $\neq 0$

$$= \sum_{m=0}^{\infty} \frac{\hat{b}^{2m}}{(2m)!} \int_{-\infty}^{\infty} dx\, x^{2m} e^{-\frac{a}{2}x^2} =$$

$\left\{ \begin{matrix} \text{G \& R} \\ \text{FI II 743/337} \end{matrix} \right\}$

$$= \sum_{m=0}^{\infty} \frac{\hat{b}^{2m}}{(2m)!} \frac{(2m-1)!!}{a^m} \sqrt{\frac{2\pi}{a}} =$$

$$= \sqrt{\frac{2\pi}{a}} \sum_{m=0}^{\infty} \left(\frac{\hat{b}^2}{a}\right)^m \frac{(2m-1)!!}{(2m)!} =$$

$$= \sqrt{\frac{2\pi}{a}} \sum_{m=0}^{\infty} \left(\frac{\hat{b}^2}{2a}\right)^m \frac{1}{m!} =$$

$$= \sqrt{\frac{2\pi}{a}}\; e^{\frac{\hat{b}^2}{2a}} = L$$

$\square$

$$\frac{(2m-1)!!}{(2m)!} = \frac{(2m+1)!!}{(2m)!(2m+1)} = \frac{(2m+1)!!}{(2m+1)!} = \frac{1}{2^m m!}$$

$$(2n)!! = 2^n n!$$

$$(2n+1)!! = \frac{(2n+1)!}{(2n)!!} = \frac{(2n+1)!}{2^n n!}$$

$$8!! = 2\cdot 4\cdot 6\cdot 8$$

$$9!! = 1\cdot 3\cdot 5\cdot 7\cdot 9$$

To compute the functional determinant we solve the differential (eigen) equation

$$\left[\partial_\tau - \mu - \frac{u}{2} + \varepsilon - iu\,\phi(\tau)\right] f_n(\tau) = \alpha_n f_n(\tau)$$

with APBC $\qquad f_n(0) = -f_n(\beta)$.

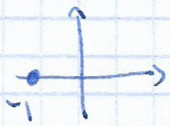
The solution reads

$$f_n(\tau) = C\, e^{\int_0^\tau d\rho \left[\mu + \frac{u}{2} - \varepsilon + iu\,\phi(\rho) + \alpha_n\right]}$$

C A constant

APBC

$$f_n(0) = C = -f_n(\beta) = -C\, e^{\int_0^\beta d\tau \left[\mu + \frac{u}{2} - \varepsilon + iu\,\phi(\tau) + \alpha_n\right]}$$



$$-1 = e^{-(2n+1)\pi i} = e^{\beta\left(\mu + \frac{u}{2} - \varepsilon + \alpha_n\right) + iu\int_0^\beta d\tau\, \phi(\tau)}$$

$$\Rightarrow \qquad \alpha_n = -i \underbrace{\frac{2n+1}{\beta}\pi}_{\omega_n} - \mu - \frac{u}{2} + \varepsilon - i\frac{u}{\beta}\int_0^\beta d\tau\, \phi(\tau)$$

$$\boxed{\alpha_n = -i\omega_n - \mu - \frac{u}{2} + \varepsilon - i\frac{u}{\beta}\int_0^\beta d\tau\, \phi(\tau)}$$

$$\mathrm{Det}\left[\partial_\tau - h\right] = \mathrm{Det}\begin{bmatrix} \alpha_1 & & \\ & \ddots & \\ & & \alpha_{\cdots} \end{bmatrix} = \prod_n \alpha_n = e^{\sum_n \ln \alpha_n}$$

Hence, we obtain

$$\boxed{S_{eff} = -2\sum_{\omega_n} \ln\left[-i\omega_n - \mu - \frac{U}{2} + \varepsilon - i\frac{U}{\beta}\int_0^\beta d\tau\, \phi(\tau)\right] + \frac{U}{2}\int_0^\beta d\tau\, \phi(\tau)^2}$$

$$Z = \int D[\phi]\, e^{-S_{eff}}$$

Using results from the lecture for non-interacting fermions $\xrightarrow{\beta \to 0}$

$$Z = N\int D[\phi] \left(1 + e^{\beta(\mu + \frac{U}{2} - \varepsilon)} e^{iU\int_0^\beta d\tau\, \phi(\tau)}\right)^2 e^{-\frac{U}{2}\int_0^\beta d\tau\, \phi^2(\tau)}$$

$$Z = N\int D[\phi]\left(1 + 2e^{\beta(\mu + \frac{U}{2} - \varepsilon)} e^{iU\int_0^\beta d\tau\, \phi(\tau)} + e^{\beta(2\mu - 2\varepsilon + U)} e^{i2U\int_0^\beta d\tau\, \phi(\tau)}\right) e^{-\frac{U}{2}\int_0^\beta d\tau\, \phi^2(\tau)}$$

$$= N\int D[\phi]\left(e^{-\frac{U}{2}\int_0^\beta d\tau\, \phi(\tau)^2} + 2e^{\beta(\mu + \frac{U}{2} - \varepsilon)} e^{-\frac{U}{2}\int_0^\beta d\tau\, \phi(\tau)^2 + iU\int_0^\beta d\tau\, \phi(\tau)} + e^{\beta(2\mu - 2\varepsilon + U)} e^{-\frac{U}{2}\int_0^\beta d\tau\, \phi(\tau)^2 + 2iU\int_0^\beta d\tau\, \phi(\tau)}\right) =$$

$$= 1 + 2e^{\beta(\mu + \frac{U}{2} - \varepsilon)} e^{\beta\frac{(iU)^2}{2U}} + e^{\beta(2\mu - 2\varepsilon + U)} e^{\frac{(2iU)^2}{2U}\beta} =$$

$$= 1 + 2e^{\beta(\mu + \frac{U}{2} - \varepsilon)} e^{-\frac{U^2}{2U}\beta} + e^{\beta(2\mu - 2\varepsilon + U)} e^{-2\frac{U^2}{U}\beta} =$$

$$= 1 + 2e^{\beta(\mu - \varepsilon)} e^{\beta\frac{U}{2} - \frac{U}{2}\beta} + e^{\beta(2\mu - 2\varepsilon)} e^{\beta U} e^{-2U\beta} =$$

$$= 1 + 2e^{\beta(\mu - \varepsilon)} + e^{2\beta(\mu - \varepsilon)} e^{-\beta U} =$$

$$\boxed{= 1 + 2e^{-\beta(\varepsilon - \mu)} + (e^{-\beta(\varepsilon - \mu)})^2 e^{-\beta U}}$$

$$\int d\mu(\xi)\, e^{-\sum_{\alpha\beta} \xi_\alpha^* H_{\alpha\beta} \xi_\beta + \sum_\alpha (\eta_\alpha^* \xi_\alpha + \eta_\alpha \xi_\alpha^*)} =$$

$$= [\operatorname{Det} H]^{-\zeta}\, e^{\sum_{\alpha\beta} \eta_\alpha^* H^{-1}_{\alpha\beta} \eta_\beta}$$

$$d\mu(\xi) = \frac{1}{\mathcal{N}} \prod_\alpha d\xi_\alpha^* \, d\xi_\alpha$$

$$\mathcal{N} = \begin{cases} 2\pi i & \text{bosons} \\ 1 & \text{fermions} \end{cases} \qquad \zeta = \begin{cases} 1 & \text{bosons} \\ -1 & \text{fermions} \end{cases}$$

$$Z = \operatorname{Tr} e^{-\beta(\hat{H} - \mu \hat{N})} = \lim_{M\to\infty} \int \prod_{k=1}^{M} \prod_\alpha \frac{1}{\mathcal{N}} d\phi_{\alpha,k}^* \, d\phi_{\alpha,k}\, e^{-S[\phi^*,\phi]}$$

$$\phi_{\alpha,0} = \zeta \phi_{\alpha,M}$$

$$S[\phi^*,\phi] = \varepsilon \sum_{k=2}^{M} \left[\sum_\alpha \phi_{\alpha k}^* \left\{ \frac{\phi_{\alpha k} - \phi_{\alpha k-1}}{\varepsilon} - \mu \phi_{\alpha k-1} \right\} + H(\phi_{\alpha k}^* \phi_{\alpha k-1}) \right]$$

$$+ \varepsilon \left[\sum_\alpha \phi_{\alpha 1}^* \left\{ \frac{\phi_{\alpha 1} - \zeta \phi_{\alpha M}}{\varepsilon} - \mu \zeta \phi_{\alpha M} \right\} + H(\phi_{\alpha 1}^*, \zeta \phi_{\alpha M}) \right]$$

$$Z = \int_{\phi_\alpha(\beta) = \zeta \phi_\alpha(0)} D[\phi^* \phi]\, e^{-\int_0^\beta d\tau \left\{ \sum_\alpha \phi_\alpha^*(\tau)(\partial_\tau - \mu)\phi_\alpha(\tau) + H(\phi_\alpha^*(\tau), \phi_\alpha(\tau)) \right\}}$$

Discrete limit

$$H_0 = \sum_\alpha \varepsilon_\alpha a_\alpha^\dagger a_\alpha$$

$$Z_0 = \lim_{M\to\infty} \prod_\alpha \left[\prod_{k=1}^{M} \int \frac{1}{\mathcal{N}}\, d\phi_{k\alpha}^* \, d\phi_{k\alpha} \, e^{-\sum_{j,k=1}^{M} \phi_{j\alpha}^* S_{jk}^{\alpha} \phi_{k\alpha}} \right] =$$

$$= \lim_{M\to\infty} \prod_\alpha \left[\operatorname{Det} S^{(\alpha)} \right]^{-\zeta}$$

$$S^{(\alpha)} = \begin{pmatrix} 1 & 0 & 0 & \cdots & & -\zeta a \\ -a & 1 & 0 & & & \vdots \\ 0 & -a & 1 & & & \vdots \\ 0 & 0 & -a & & 0 & \vdots \\ \vdots & \vdots & 0 & & 1 & 0 \\ \vdots & \vdots & \vdots & & -a & 1 \end{pmatrix} \qquad \phi_\alpha = \begin{pmatrix} \phi_{1\alpha} \\ \phi_{2\alpha} \\ \vdots \\ \vdots \\ \phi_{M\alpha} \end{pmatrix}$$

$$a = 1 - \frac{\beta}{M}(\varepsilon_\alpha - \mu)$$

$$\operatorname{Det} S^\alpha = \begin{vmatrix} 1 & 0 & \cdots & \\ -a & 1 & & \vdots \\ \vdots & -a & & \vdots \\ \vdots & \vdots & & -a \end{vmatrix}_{M-1} + a \begin{vmatrix} 0 & 0 & \cdots & & -\zeta a \\ -a & 1 & & & \\ 0 & -a & & & \\ & & & & \vdots \\ & & & -a & 1 \end{vmatrix}_{M-1} =$$

$$= 1 - \zeta a^2 (-1)^{M-1} \cdot \begin{vmatrix} -a & 1 & \\ 0 & -a & 1 \\ & & \ddots \end{vmatrix}_{M-2} = 1 + \zeta (-1)^{M-1} (-a)^{M}$$

$$Z_0 = \lim_{M\to\infty} \prod_\alpha \left[1 + (-1)^{M-1} \zeta (-a)^M \right] =$$

$$= \lim_{M\to\infty} \prod_\alpha \left[1 - \zeta \left(1 - \frac{\beta(\varepsilon_\alpha - \mu)}{M} \right)^M \right] =$$

$$= \prod_\alpha \left[1 - \zeta e^{-\beta(\varepsilon_\alpha - \mu)} \right]$$

continuum limit I

$$\psi_\alpha(\tau) = \frac{1}{\sqrt{\beta}}\sum_{\omega_n} e^{-i\omega_n\tau}\psi_{n\alpha}$$

$$\psi^*_\alpha(\tau) = \frac{1}{\sqrt{\beta}}\sum_{\omega_n} e^{i\omega_n\tau}\psi^*_{n\alpha}$$

$$Z_0 = \int D[\psi(\tau)]\, e^{-S_0[\psi]}$$

$$S_0[\psi] = \int_0^\beta d\tau \sum_\alpha \psi^*_\alpha(\tau)\,(\partial_\tau + \xi_\alpha)\,\psi_\alpha(\tau) =$$

$\xi_\alpha = \varepsilon_\alpha - \mu$

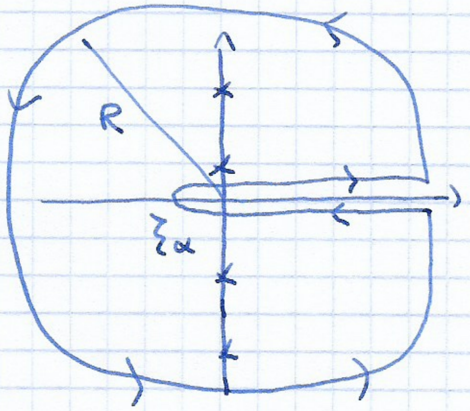
$$= \sum_\alpha \sum_{\omega_n} \psi^*_{\alpha n}(-i\omega_n + \xi_\alpha)\psi_{\alpha n}$$

$$D[\psi] = \prod_\alpha \prod_{\omega_n} d\psi^*_{n\alpha}\, d\psi_{n\alpha}$$

$$Z_0 = \prod_\alpha \prod_{\omega_n}(-i\omega_n + \xi_\alpha) = e^{\sum_\alpha \sum_{\omega_n} \ln(-i\omega_n + \xi_\alpha)}$$

$$n_F = \frac{1}{e^{\beta z}+1}$$

$$\lim_{\eta\to 0^+}\sum_{\omega_n} \ln(-i\omega_n + \xi_\alpha)e^{i\omega_n\eta} = -\lim_{\eta\to 0^+}\beta\oint\frac{dz}{2\pi i}\, n_F(z)\ln(-z+\xi_\alpha)e^{z\eta} =$$



factor $e^{z\eta}$ ensures that the contribution from C in $R\to\infty$ vanishes

$$= -\beta\left(\int_{\xi_\alpha-\eta}^{\infty}\frac{d\varepsilon}{2\pi i}\, n_F(\varepsilon)\ln(-\varepsilon - i\eta + \xi_\alpha) + \int_{\infty}^{\xi_\alpha-\eta}\frac{d\varepsilon}{2\pi i}\, n_F(\varepsilon)\ln(-\varepsilon + i\eta + \xi_\alpha)\right) =$$

$$= -\beta\left(\int_{\xi_\alpha-\eta}^{\infty}\frac{d\varepsilon}{2\pi i}\left(n_F(\varepsilon)\ln(-\varepsilon - i\eta + \xi_\alpha) + \text{h.c.}\right)\right)$$

using

$$n_F(\varepsilon) = -\frac{1}{\beta}\frac{d}{d\varepsilon}\ln(1+e^{-\beta\varepsilon})$$

and integrating by parts

$$= \int_{\xi_\alpha - i\eta}^{\infty} \frac{d\varepsilon}{2\pi i} \ln\left|1+e^{-\beta\varepsilon}\right| \frac{d}{d\varepsilon}\left[\ln(-\varepsilon - i\eta + \xi_\alpha) + h.c.\right] =$$

$$= \int_{\xi_\alpha - i\eta}^{\infty} \frac{d\varepsilon}{2\pi i} \ln\left|1+e^{-\beta\varepsilon}\right| \left[-2\pi i\, \delta(\varepsilon - \xi_\alpha)\right] =$$

$$= \ln\left(1+e^{-\beta\xi_\alpha}\right)$$

$$\Rightarrow \quad Z_0 = \prod_\alpha \left(1+e^{-\beta\xi_\alpha}\right)$$

Continuum limit II

$$e^{\sum_{\omega_n} \ln(-i\omega_n + \xi_\alpha)} \to e^{\sum_{\omega_n} \ln\left(\frac{-i\omega_n + \xi_\alpha}{-i\omega_n}\right)}$$

Note that

1) $$\int_0^{\xi_\alpha} \frac{dx}{x - i\omega_n} = \ln\left(\frac{-i\omega_n + \xi_\alpha}{-i\omega_n}\right)$$

2) $$\sum_{\omega_n} \frac{e^{i\omega_n \eta}}{i\omega_n - x} = -\frac{\beta}{2\pi i} \oint \frac{dz}{e^{\beta z}+1} \frac{e^{\eta z}}{z - x} = -\frac{\beta}{e^{\beta x}+1}$$

3) $$\sum_n \ln\left(\frac{-i\omega_n + \xi_\alpha}{-i\omega_n}\right) = -\int_0^{\xi_\alpha} dx \sum_{\omega_n} \frac{1}{i\omega_n - x} =$$

$$= \int_0^{\xi_\alpha} dx \frac{\beta}{e^{\beta x}+1} = -\beta \frac{1}{\beta} \ln\left|\frac{e^{\beta x}}{e^{\beta x}+1}\right|_0^{\xi_\alpha} = \ln\left|\frac{1}{1+e^{-\beta\xi_\alpha}}\right|$$

$$Z_0 = \prod_\alpha \left(1 + e^{-\beta \xi_\alpha}\right)$$