§5. GAS OF IDEAL FERMIONS AT LOW T

•) In general the grand canonical ensemble average is

average

Fermi-Dirac distribution function

$$\langle x \rangle = 2 \sum_{\vec{k}} x_{\vec{k}}\, f(\varepsilon_{\vec{k}}, \mu, T)$$

spin

some quantum numbers

observable

$\vec{k}$ - wave vector

If $x_{\vec{k}} = x(\varepsilon_{\vec{k}})$

function of $\varepsilon_{\vec{k}}$ not of $\vec{k}$

then

$$\boxed{\langle x \rangle} = \int d\varepsilon \; 2 \sum_{\vec{k}} \delta(\varepsilon - \varepsilon_{\vec{k}})\, f(\varepsilon, \mu, T)\, x(\varepsilon) =$$

$$\boxed{= \int d\varepsilon \; \rho(\varepsilon)\, f(\varepsilon, \mu, T)\, x(\varepsilon)}$$

$$\boxed{\rho(\varepsilon) = 2 \sum_{\vec{k}} \delta(\varepsilon - \varepsilon_{\vec{k}})}$$ - the density of states.

example $\varepsilon_{\vec{k}} = \frac{\hbar^2 \vec{k}^2}{2m}, \quad d = 3$

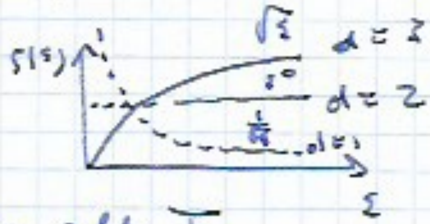
$$\langle x \rangle = 2 \sum_{k_x} \sum_{k_y} \sum_{k_z} x(\varepsilon_k)\, f(\varepsilon_{\vec{k}}) =$$

$$= 2 \frac{V}{(2\pi)^3} \int d^3k \; x(\varepsilon_k)\, f(\varepsilon_k) = \left\{ \begin{array}{l} \text{spherical} \\ \text{coordinates} \end{array} \right\} =$$

$$= 2 \frac{V}{(2\pi)^3} 4\pi \int_0^\infty dk\, k^2 \, x(\varepsilon_k)\, f(\varepsilon_k) = \left\{ \begin{array}{l} \varepsilon = \frac{\hbar^2 k^2}{2m} \\ d\varepsilon = \frac{\hbar^2}{m} k\, dk \end{array} \right\} =$$

$$= \underbrace{\frac{V}{2\pi^2} \int_0^\infty d\varepsilon \; \frac{2m}{\hbar^2} \sqrt{\frac{2m\varepsilon}{\hbar^2}}}_{\rho(\varepsilon)} \; x(\varepsilon)\, f(\varepsilon)$$

$$\boxed{\rho(\varepsilon) = \frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} \varepsilon^{1/2}}$$ DOS



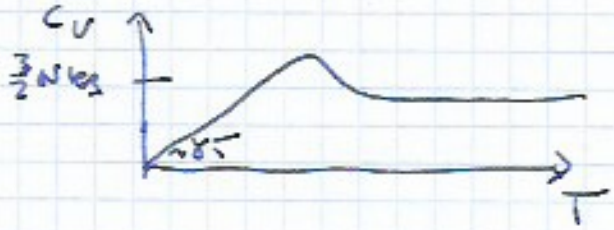
- <u>Specific heat of electrons</u>

classically $c_V = \frac{3}{2} N k_B$ for all T

experimentally $c_V \simeq \gamma T$ at low T

$C_V = \left(\frac{\partial U}{\partial T}\right)_V$ heat capacity

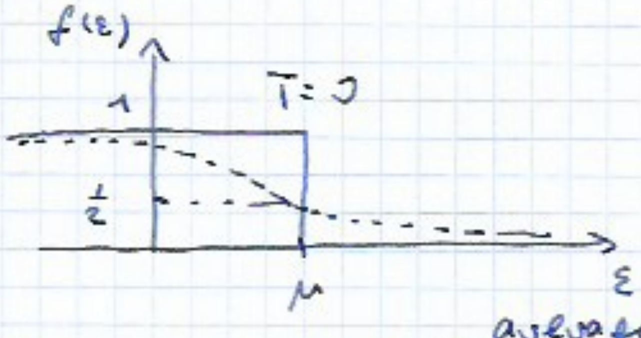
$c_V = \frac{C_V}{V}$ specific heat



At low T we need

$$\Delta U = U(T) - U(0)$$ internal energy difference

$$\Delta U = \int_0^\infty d\varepsilon\, \rho(\varepsilon) f(\varepsilon)\, \varepsilon - \int_0^{\varepsilon_F} d\varepsilon\, \rho(\varepsilon)\, \varepsilon$$



$f(\varepsilon) \xrightarrow[T\to 0]{} \theta(\mu - \varepsilon)$

$\mu(T=0) = \varepsilon_F$

Letting a constant $\overbrace{\text{number}}^{\text{average}}$ of particles

$$N = \int_0^\infty d\varepsilon\, \rho(\varepsilon) f(\varepsilon) = \int_0^{\varepsilon_F} d\varepsilon\, \rho(\varepsilon) \quad \Big| \cdot \varepsilon_F$$

hence

$$\left(\int_0^{\varepsilon_F} d\varepsilon + \int_{\varepsilon_F}^{\infty} d\varepsilon\right) \varepsilon_F\, \rho(\varepsilon) f(\varepsilon) = \int_0^{\varepsilon_F} d\varepsilon\, \varepsilon_F\, \rho(\varepsilon) \qquad (*)$$

And we get the ΔU equation

$$\boxed{\Delta U = \int_{\varepsilon_F}^{\infty} d\varepsilon\,(\varepsilon-\varepsilon_F)\,f(\varepsilon)\,\rho(\varepsilon) + \int_0^{\varepsilon_F} d\varepsilon\,(\varepsilon_F-\varepsilon)\,[1-f(\varepsilon)]\,\rho(\varepsilon)}$$

Proof:

$$\underline{\int_{\varepsilon_F}^{\infty} d\varepsilon\, \varepsilon f \rho} - \overbrace{\int_{\varepsilon_F}^{\infty} \varepsilon_F f \rho + \int_0^{\varepsilon_F} d\varepsilon\, \varepsilon_F \rho - \int_0^{\varepsilon_F} d\varepsilon\, \varepsilon_F f \rho}^{=0 \ (*)} -$$

$$- \underline{\underline{\int_0^{\varepsilon_F} d\varepsilon\, \varepsilon \rho}} + \underline{\int_0^{\varepsilon_F} d\varepsilon\, \varepsilon f \rho} =$$

$$= \int_0^{\infty} d\varepsilon\, \varepsilon \rho f - \int_0^{\varepsilon_F} \varepsilon \rho$$

D

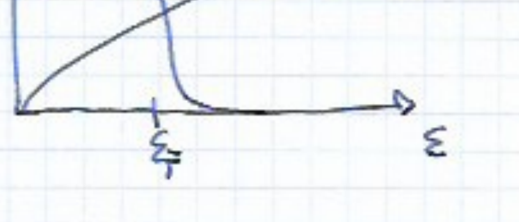
Interpretation:

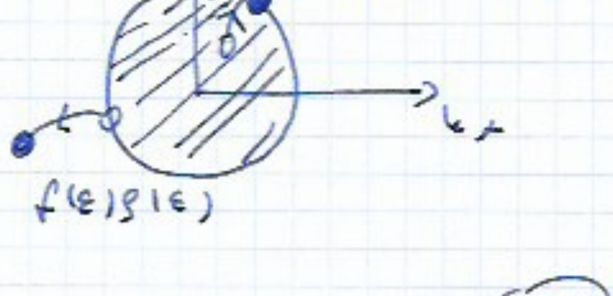
$\int_{\varepsilon_F}^{\infty} d\varepsilon\,(\varepsilon-\varepsilon_F)\,f(\varepsilon)\,\rho(\varepsilon)$ – energy needed to excite an electron $\varepsilon_F \to \varepsilon > \varepsilon_F$

$\int_0^{\varepsilon_F} d\varepsilon\,(\varepsilon_F-\varepsilon)\,f(\varepsilon)\,\rho(\varepsilon)$ – energy needed to excited an electron $\varepsilon \to \varepsilon_F$ $\quad (\varepsilon < \varepsilon_F)$

$\rho(\varepsilon) f(\varepsilon)\, d\varepsilon$ = # of electrons at $[\varepsilon, \varepsilon + d\varepsilon]$

$[1 - f(\varepsilon)]$ – probability of removing an electron from $\varepsilon < \varepsilon_F$





Specific heat (only f - depends on T)

$$c_V = \left(\frac{dU}{dT}\right)_V = \int_{\varepsilon_F}^{\infty} d\varepsilon\,(\varepsilon-\varepsilon_F)\left(\frac{df}{dT}\right)\rho(\varepsilon) - \int_0^{\varepsilon_F} d\varepsilon\,(\varepsilon_F-\varepsilon)\left(\frac{df}{dT}\right)\rho(\varepsilon)$$

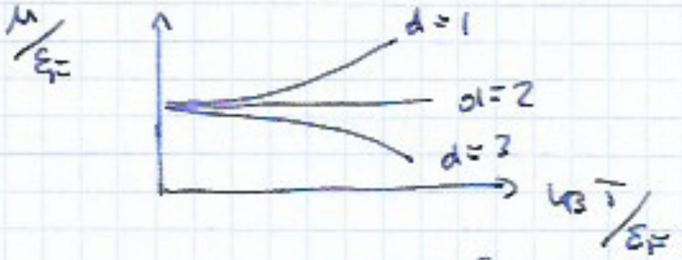
$$= \int_0^{\infty} d\varepsilon\,(\varepsilon-\varepsilon_F)\left(\frac{df}{dT}\right)\rho(\varepsilon)$$

typically $\frac{\varepsilon_F}{k_B} \simeq 50\,000\,K$!

At low $T \ll \varepsilon_F$, $\rho(\varepsilon)$ weakly changes with ε, so

$$c_V \simeq \rho(\varepsilon_F)\int_0^{\infty} d\varepsilon\,(\varepsilon-\varepsilon_F)\left(\frac{df}{dT}\right)$$

observing that $\mu(T) \simeq \varepsilon_F$ at low T



$$\left(\frac{df}{d\varepsilon}\right) = \frac{\varepsilon-\varepsilon_F}{k_BT^2}\,\frac{e^{\frac{\varepsilon-\varepsilon_F}{k_BT}}}{\left(e^{\frac{\varepsilon-\varepsilon_F}{k_BT}}+1\right)^2}$$

changing the integration variable $\left\{ x = \frac{\varepsilon-\varepsilon_F}{k_BT} \right\}$

$$c_V = k_B^2\,T\,\rho(\varepsilon_F)\int_{-\varepsilon_F/k_BT}^{\infty} dx\; x^2\,\frac{e^x}{(e^x+1)^2} \simeq k_B^2\,T\rho(\varepsilon_F)\int_{-\infty}^{\infty} dx\,\frac{x^2 e^x}{(e^x+1)^2}$$

$$-\varepsilon_F/k_BT \xrightarrow{T\to 0} -\infty$$

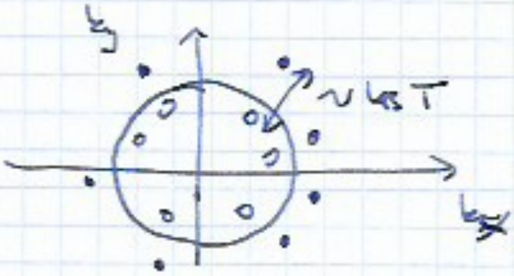
we know that

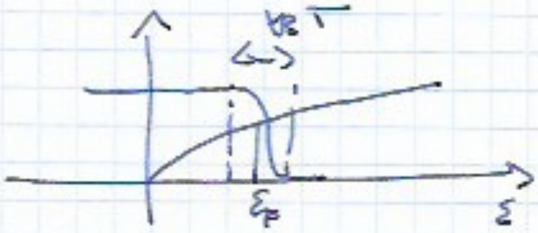
$$\int_{-\infty}^{\infty} dx\,\frac{x^2 e^x}{(e^x+1)^2} = \pi^2/3$$

Finally

$$\boxed{C_V = \frac{1}{3}\pi^2 \rho(\varepsilon_F) k_B^2 T = \gamma T}$$

The number of excited electrons around ε_F rises linearly with T





a degenerate fermion limit

Measurement of γ give rise to information about the <u>effective mass m^*</u>

$$\gamma = \frac{1}{3}\pi^2 \frac{1}{2\pi^2}\left(\frac{2m^*}{\hbar^2}\right)^{3/2} \varepsilon_F^{1/2}$$

$$\varepsilon_F = \frac{\hbar^2}{2m^*}(3\pi^2 n)^{2/3}$$

so

$$\gamma = \frac{1}{6}\frac{2m^*}{\hbar^2}(3\pi^2 n)^{1/3} \longleftrightarrow \boxed{\gamma \sim m^*}$$

$$\boxed{m^* = \frac{3\gamma\hbar^2}{(3\pi^2 n)^{1/3}}} \qquad \boxed{\frac{1}{2} m^* v_F^2 = \varepsilon_F}$$

• liquid ^3He $\quad 2e + 2p + 1n \rightarrow$ spin $\hbar/2$

an atom mass

$$m^*_{^3He} \simeq 2,8\, m_{^3He}$$

$$c_V \sim \gamma_{^3He} T$$

• alkali atoms K, Na, Ca

transition metals

an electron mass

$$m^* \sim 1-10\, m_e$$

• heavy fermions

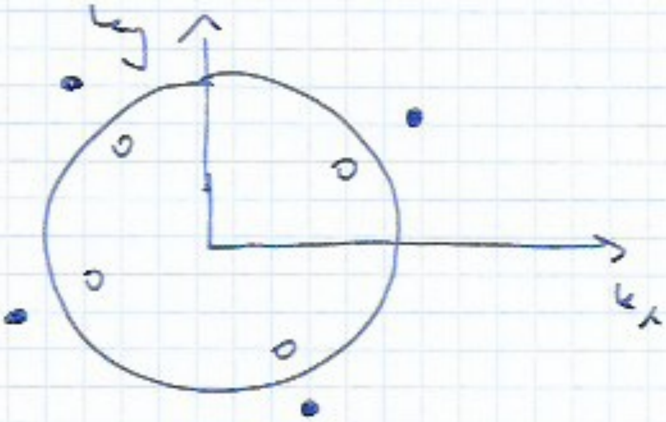
UBe_{13}, UCd_{11}, UPt_3, UCu_5, ...

$$m^* \sim 100 - 1000\, m_e$$

§6. LANDAU QUASIPARTICLES

$m^* \neq m_0 \Rightarrow$ the interaction modifies dynamical characteristics of the system but the Fermi fermionic properties remain $\quad C_V = \gamma^* T$.

In 1956 L. Landau proposed a theory where a system of interacting electrons is viewed as a system of <u>weakly</u> interacting <u>quasiparticles</u> with Fermi-Dirac distribution function.



Small numbers of electrons above FS and holes below FS

$\Downarrow$

the scattering is rare

Let $\quad n_{\vec{k}\sigma}$ - occupation of $|\vec{k}\sigma\rangle$ at $T>0$

$\quad\quad n^0_{\vec{k}\sigma}$ - occupation of $|\vec{k}\sigma\rangle$ at $T=0$

At low T the energy difference must be small

$$\boxed{\Delta u} = u(T) - u(0) = \sum_{\vec{k}\sigma} \varepsilon_{\vec{k}} n_{\vec{k}\sigma} - \sum_{\vec{k}\sigma} \varepsilon_{\vec{k}\sigma} n^0_{\vec{k}\sigma} =$$

$$= \boxed{\sum_{\vec{k}\sigma} \varepsilon_{\vec{k}} \delta n_{\vec{k}\sigma}} \quad \text{a type of Taylor expansion}$$

$\delta n_{\vec{k}\sigma} = n_{\vec{k}\sigma} - n^0_{\vec{k}\sigma} \leftarrow$ small if <u>$k_B T \ll \varepsilon_F$</u>

The real interaction between the quasiparticles can be included in the expansion terms of higher order

$$\boxed{\delta U = \sum_{\vec{k}\sigma} \varepsilon_{\vec{k}} \, \delta n_{\vec{k}\sigma} + \frac{1}{2} \sum_{\vec{k}\vec{k}'} \sum_{\sigma\sigma'} f_{\vec{k}\vec{k}'}^{\sigma\sigma'} \, \delta n_{\vec{k}\sigma} \, \delta n_{\vec{k}'\sigma'} + \dots}$$

This is a standard model of the <u>Fermi liquid</u>

Define a dispersion relation

$$\boxed{E_{\vec{k}\sigma} = \varepsilon_{\vec{k}} + \frac{1}{2} \sum_{\vec{k}'\sigma'} f_{\vec{k}\vec{k}'}^{\sigma\sigma'} \, \delta n_{\vec{k}'\sigma'}}$$

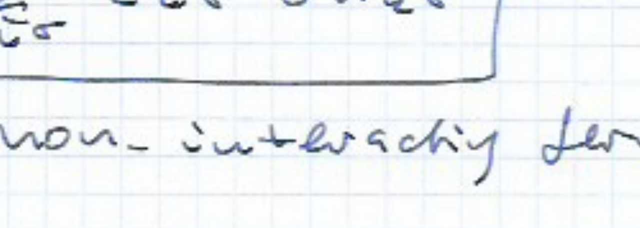
Then $\boxed{\delta U = \sum_{\vec{k}\sigma} E_{\vec{k}\sigma} \, \delta n_{\vec{k}\sigma}}$

similarly to non-interacting fermions.

For non-interacting fermions, close to k_F we write

$$\varepsilon_{\vec{k}} \simeq \varepsilon_F + \left.\frac{\partial \varepsilon_{\vec{k}}}{\partial \vec{k}}\right|_{k_F} \cdot (\vec{k} - \vec{k}_F) \simeq \varepsilon_F + \vec{v}_F \cdot (\vec{k} - \vec{k}_F)$$

$$\vec{v}_F = \frac{\vec{p}_F}{m_0} = \frac{\hbar \vec{k}_F}{m_0} \quad \text{- Fermi velocity}$$



linearized dispersion relation

For interacting particles we expand similarly

$$E_{\vec{k}\sigma} = \left(1 + \frac{1}{2\xi_{\vec{k}}}\sum_{\vec{k}'\sigma'} f_{\vec{k}\vec{k}'}^{\sigma\sigma'}\,\delta n_{\vec{k}'\sigma'}\right)\xi_{\vec{k}} =$$

$$E_{\vec{k}\sigma} = \varepsilon_F^* + \frac{\hbar k_F}{m^*}(k - k_F)$$

$$= Z_{\vec{k}\sigma}\,\xi_{\vec{k}}$$

$$\xi_{\vec{k}} = \varepsilon_F + \vec{v}_F\cdot(\vec{k}-\vec{k}_F)$$

quasiparticle renormalization factor

$$Z_{\vec{k}\sigma} = \varepsilon_F^* + \left(1 + \frac{1}{2\xi_F}\sum_{\vec{k}'\sigma'} f_{\vec{k}\vec{k}'}^{\sigma\sigma'}\,\delta n_{\vec{k}'\sigma'}\right)\frac{\hbar k_F}{m_0}$$

Here,

$$\boxed{\frac{1}{m^*_\sigma} = \left(1 + \frac{1}{2\xi_F}\sum_{\vec{k}'\sigma'} f_{\vec{k}\vec{k}'}^{\sigma\sigma'}\,\delta n_{\vec{k}'\sigma'}\right)\frac{1}{m_0}}$$

Applying the statistical physics to the quasiparticles with effective mass we can qualitatively explain properties of metals, alloys, liquid ^{3}He, neutron stars or white dwarfs.

Some basic results – renormalization

specific heat

$$C_V = \gamma T = \frac{m^*}{m_0}\gamma_0 T \qquad \frac{m^*}{m_0} = 1 + \frac{1}{3}F_1^s$$

Pauli susceptibility

$$\chi = \chi_0 \frac{1}{1+F_0^a}$$

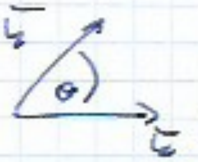
Compressibility

$$K = -\frac{1}{V}\left(\frac{\partial V}{\partial p}\right) = \frac{1}{n}\left(\frac{\partial n}{\partial \mu}\right) = K_0\,(1+F_0^s)$$

sound speed

$$c_s = c_s^0\,\frac{1+F_0^s}{1+\frac{1}{3}F_1^s}$$

where for isotropic system



$$f_{\vec{k}\vec{k}'}^{\sigma\sigma'} = f^{\sigma\sigma'}(\cos\theta) = \sum_{l=0}^{\infty} f_l^{\sigma\sigma'}\, P_l(\cos\theta)$$

$$f_l^{s(a)} = \frac{2l+1}{2}\int_{-1}^{1} dx\, P_l(x)\,\frac{f^{\uparrow\uparrow}(x) \pm f^{\uparrow\downarrow}(x)}{2}$$

$$\boxed{F_l^{s(a)} = \rho(\varepsilon_F)\, f_l^{s(a)}}$$ Landau parameters

Methods of quantum field theory and density functional theory relate

$$F_l^{s(a)} \longleftrightarrow \frac{e^2}{|\vec{r}-\vec{r}'|}$$ bare Coulomb interaction.