§7. BOSE-EINSTEIN CONDENSATION

1926 A. Einstein predicted that at $T < T_c$ the lowest energy level is occupied by macroscopic number of bosons

$$n(\varepsilon_{\vec k}) \sim \mathcal{O}(1) \quad \text{for } T > T_c$$

$$n(\varepsilon_k) \sim \begin{cases} \mathcal{O}(1) & k > 0 \\ \mathcal{O}(N) & k = 0 \end{cases} \qquad \text{for } T < T_c$$

At $T = T_c$ there is a phase transition - BEC

This BEC was seen in ^4He at $T_\lambda \sim 2K$ and ^{87}Rb at $T_c \sim 10^{-8}K$

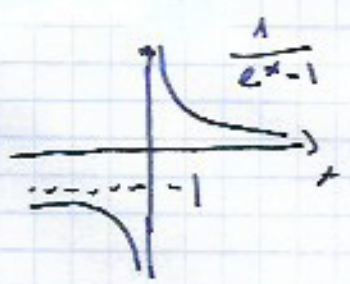
•) The chemical potential at very low temperatures

BE - distribution function

$$n_{\vec k} = f(\varepsilon_{\vec k}) = \frac{1}{e^{\beta(\varepsilon_{\vec k} - \mu)} - 1} \quad , \quad \varepsilon_{\vec k} = \frac{\hbar^2 \vec k^2}{2m}$$

it is positive as long as

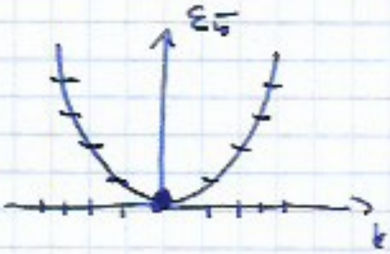
$$\mu < \varepsilon_0 = 0$$



The occupation of the lowest energy level $\vec{k}=0$

$$f(0,T) = \frac{1}{e^{-\beta\mu}-1}$$

At $T=0$ all N bosons must be in $\vec{k}=0$ state



$$\lim_{T\to 0^+} f(0,T) = N \simeq \lim_{T\to 0^+} \frac{1}{e^{-\beta\mu}-1} \simeq \frac{1}{1-\beta\mu+\dots-1} = -\frac{k_B T}{\mu}$$

hence,

$$\boxed{\mu = -\frac{k_B T}{N}}$$

and

$$\boxed{\mu \xrightarrow[T\to 0^+]{} 0^-}$$

Hence, always $\mu < 0$, otherwise the BE series

$$\tilde{\omega}_b = \prod_{\vec{k}} \left(1 + e^{-\beta(\varepsilon_{\vec{k}}-\mu)} + \left(e^{-\beta(\varepsilon_{\vec{k}}-\mu)}\right)^2 + \dots\right) = \prod_{\vec{k}} \left(\frac{1}{1-e^{-\beta(\varepsilon_{\vec{k}}-\mu)}}\right)$$

does not converge! $\qquad\qquad$ $\forall \vec{k}$ converge

Similarly, the activity $\alpha = e^{\beta\mu} \simeq 1 - \frac{1}{N}$ at $T\to 0^+$.

•) <u>Occupation of $\varepsilon_{\vec{k}}$ levels as a function of T</u>

$$N = \sum_{\vec{k}} f(\varepsilon_{\vec{k}}, T) = N_0(T) + N_{ex}(T)$$

ground state occupation — <u>condensate</u>

excited states occupations — <u>normal</u>

$$N = N_0(T) + \int_0^\infty d\varepsilon\, \underbrace{\rho(\varepsilon)}_{\text{DOS}} f(\varepsilon, T)$$

$$\rho(\varepsilon) = \frac{V}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\sqrt{\varepsilon}$$

no spin

<u>occupation of the ground state</u>

$$N_0(T) = \frac{1}{\alpha^{-1} - 1}, \qquad \alpha = e^{\beta\mu} \simeq 1 - \frac{1}{N}$$

<u>occupation of excited states</u>

$$N_{exc}(T) = \frac{V}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} \int_0^\infty d\varepsilon \frac{\sqrt{\varepsilon}}{\alpha^{-1}e^{\beta\varepsilon} - 1} = \left\{ x = \beta\varepsilon \right\} =$$

$$= \frac{V}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} (k_B T)^{3/2} \int_0^\infty dx \frac{\sqrt{x}}{\alpha^{-1}e^{x} - 1}$$

But at low T $N_0(T) \gg 1$ hence $\alpha \simeq 1$, $\mu \to 0^-$.
Using $\alpha = 1$ and

$$\int_0^\infty dx \frac{\sqrt{x}}{e^x - 1} = 1.306\sqrt{\pi}$$

we obtain

$$N_{exc}(T) = \frac{1{,}306}{4} V \left(\frac{2m}{\pi\hbar^2}\right)^{3/2} = 2{,}612\, n_Q V$$

$$n_Q = \left(\frac{m k_B T}{2\pi\hbar^2}\right)^{3/2} = \left(\frac{1}{\lambda_{dB}}\right)^3$$ – <u>quantum concentration</u>

Finally,

$$\boxed{\frac{N_{exc}(T)}{N} = 2{,}612\, n_Q \frac{V}{N} = 2{,}612 \frac{n_Q}{n} = 2{,}612\left(\frac{a_0}{\lambda_{dB}}\right)^3}$$

in <u>BEC</u> $\lambda_{dB} \gtrsim a_0 = \left(\frac{1}{n}\right)^{1/3}$

<u>$T > T_c$</u>

<u>$T < T_c$</u>

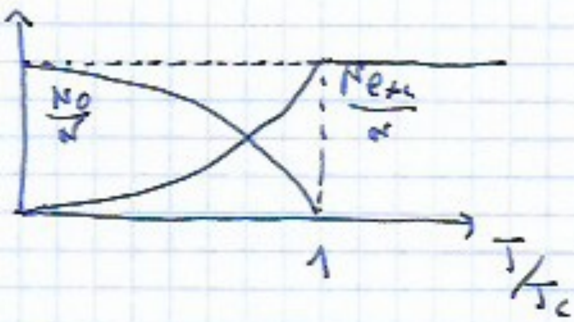
•) Condensation temperature

T_c is defined by $\alpha = 1$ and $N_{exc}(T_c) = N$

Then
$$\boxed{k_B T_c = \frac{2\pi\hbar^2}{m}\left(\frac{N}{2{,}612\,V}\right)^{2/3}}$$

Using previous result we get that for $T<T_c$

$$\boxed{\begin{aligned} \frac{N_{exc}}{N} &= \left(\frac{T}{T_c}\right)^{3/2} \\ \frac{N_0}{N} &= 1 - \left(\frac{T}{T_c}\right)^{3/2} \end{aligned}}$$



•) Properties of bosons in the condensate

$N_0 \sim O(N)$ particles are in the same quantum state $k=0$ with the same wave function

$$\varphi_{\vec{k}=0}(\vec{r}) = \frac{1}{\sqrt{V}} e^{i\vec{k}\cdot\vec{r}}\Big|_{\vec{k}=0} = \frac{1}{\sqrt{V}}$$

The ground state wave function is

$$\Psi_0(\vec{r}_1 \dots \vec{r}_{N_0}) = \varphi_0(\vec{r}_1)\dots\varphi_0(\vec{r}_{N_0}) = e^{i\sum_{i=1}^{N_0}\phi_i}\left(\frac{1}{\sqrt{V}}\right)^{N_0}$$

ϕ_i - arbitrary phase factor

Since bosons are indistinguishable, then all N_0 bosons in the condensate must have the same phase factor $\phi_i = \phi \ \forall i$

$$\Psi_0(\vec{r}_1 \dots \vec{r}_{N_0}) = e^{i N_0 N}\prod_{i=1}^{N_0}|\varphi_0(\vec{r}_i)|$$

[the same for all i

There is a macroscopic condensate of N_0 bosons, all with the same phase. Therefore, the system exhibits a phase coherence.

It is possible to use a single wave function of the condensate

$$\hat{F} = \sum_{i=1}^{N_0} \hat{f}(\vec{r}_i) \quad \text{– single particle operator}$$

$$\langle \hat{F} \rangle_{BEC} = \int \Psi_0^* \, \hat{F} \, \Psi_0 \, d_{3N_0} r =$$

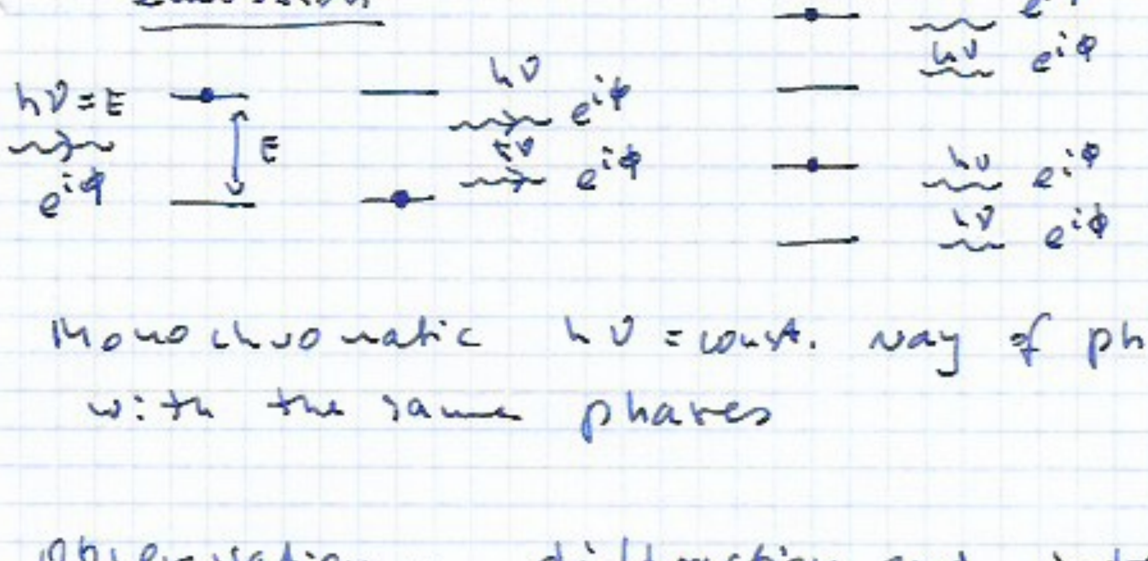
$$= N_0 \int \varphi_0^*(\vec{r}) \, \hat{f}(\vec{r}) \, \varphi_0(\vec{r})$$

$$\Psi_0(\vec{r}_1 \dots \vec{r}_{N_0}) \longrightarrow \boxed{\Psi_0(\vec{r}) = \sqrt{N_0}\, e^{i\phi} \, \varphi_0(\vec{r})}$$

BEC macroscopic wave function

•) Laser analogy

stimulated emission ⊕ photons indistinguishability



Monochromatic $h\nu = $ const. ray of photons with the same phases

Observation – diffraction and interference

•) Experimental observation of BEC

- macroscopic occupation of a single state
- phase coherence → interference

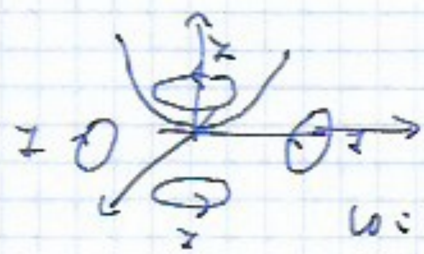
Cold atoms

$^{7}_{3}Li$	$2s^1$		$S_n = \frac{3}{2}$
$^{23}_{11}Na$	$3s^1$		$\frac{3}{2}$
$^{41}_{19}K$	$4s^1$		$\frac{3}{2}$
$^{87}_{37}Rb$	$5s^1$	$S_e = \frac{1}{2}$	$\frac{3}{2}$
$^{133}_{55}Cs$	$6s^1$		$\frac{7}{2}$
Fr	$7s^1$		

$S = 1,2,3,4 \dots$ $\boxed{\text{bosons}}$

$$\bar S = \bar S_e + \bar S_n \qquad S = |S_e - S_n|, \dots, |S_e + S_n|$$

→ magnetooptical trap



parabolic potential

$$U(\bar r) = -g\mu_B\, \bar S \cdot \bar B(\bar r)$$

coils produce magnetic fields $\bar B(\bar r)$

in vacuum

go to the center (trapping)

go away

→ cooling the system – rare gas, $r_s \sim 50 - 600\,nm$ (no crystalization)

$$n = \frac{1}{\frac{4}{3}\pi r_s^3} \sim 10^{11} - 10^{15}\,\frac{1}{cm^3}$$

1995, E. Cornell, W. Ketterle, P. Wiemann – ^{87}Rb

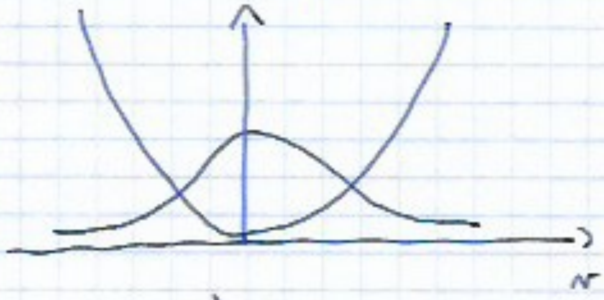
$$T_c \sim 0{,}5 - 2\ \mu K$$

$$N \sim 10^5 - 10^6 \text{ atoms}$$

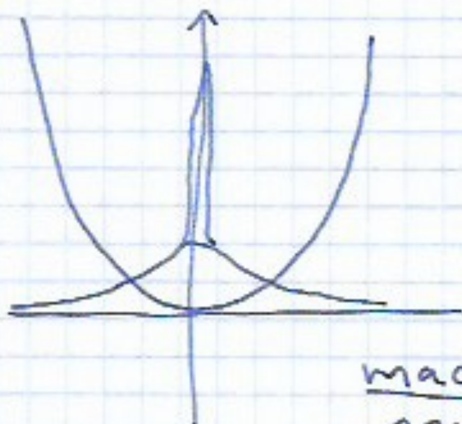
$$\frac{1}{2} m \langle v^2 \rangle = \frac{1}{2} k_B T \quad \text{- Maxwell}$$

→ <u>time of flight experiment</u>

picture / density projection



$T > T_c$

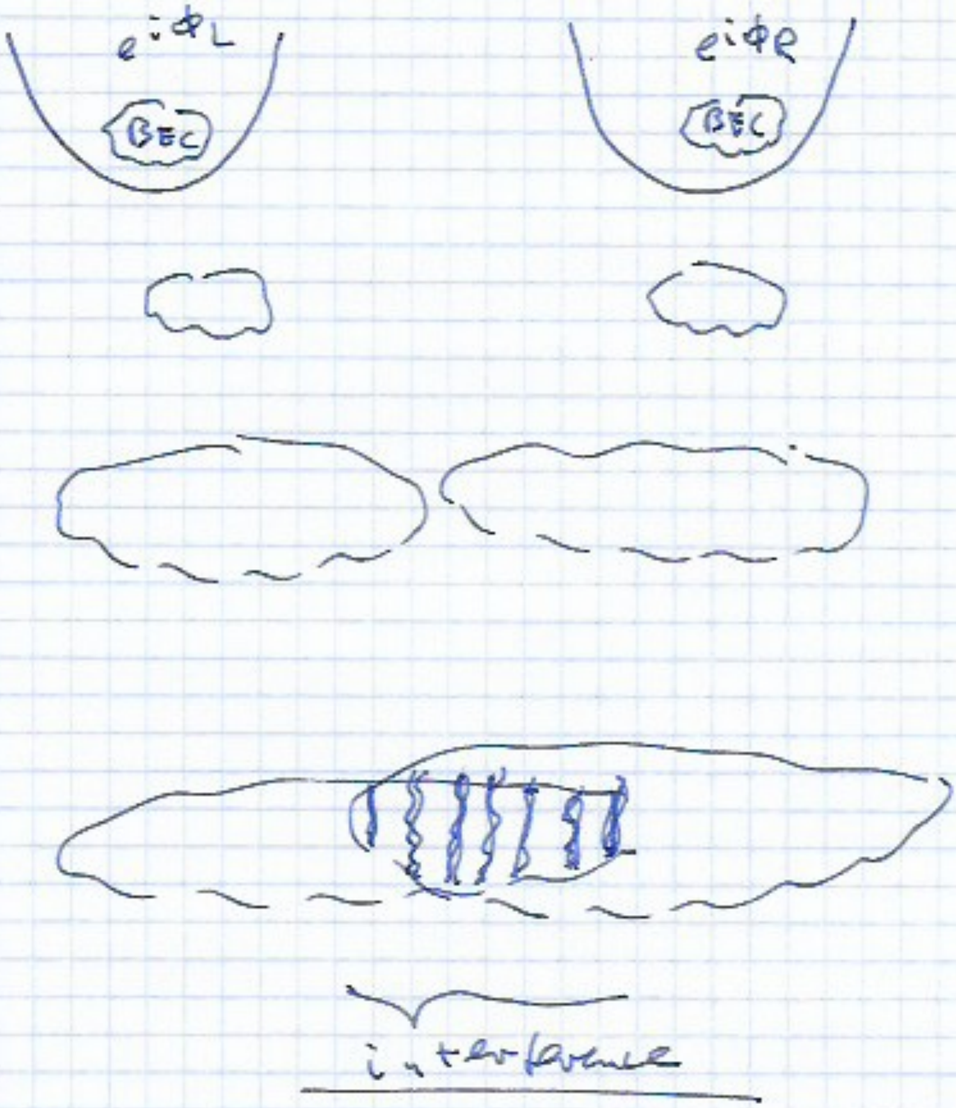


$T < T_c$

<u>macroscopic occupation</u>

	harmonic trap	uniform gas
$k_B T_c$	$\hbar \left(\frac{\omega_0^3 N}{\zeta(3)} \right)^{1/3}$	$\frac{2\pi\hbar^2}{m} \left(\frac{n}{\zeta(3/2)} \right)^{2/3}$
$\frac{N_0}{N}$	$1 - \left(T/T_c\right)^3$	$1 - \left(T/T_c\right)^{3/2}$
thermodynamic limit	$N\omega_0^3 = \text{const.}$	$\frac{N}{V} = \text{const.}$

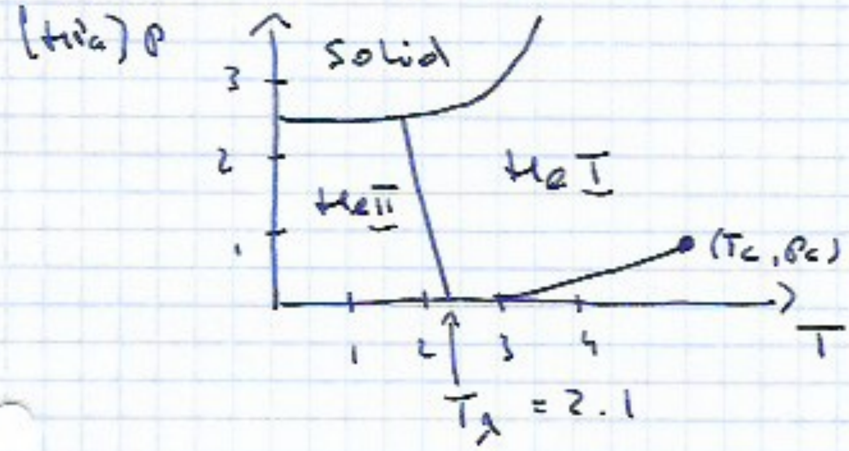
Interference patterns

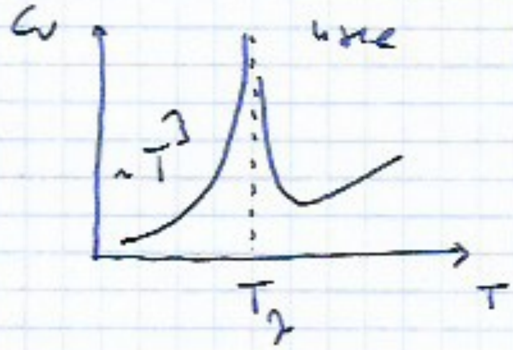


§ 8 SUPERFLUIDITY

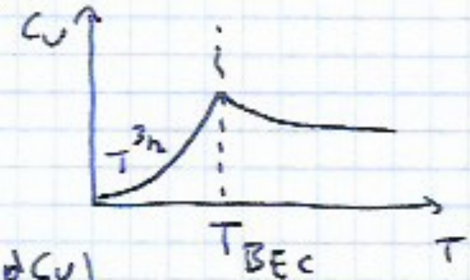
[alfredleitner.com]

Unique properties of ^4He



1) 

λ-transition



$$\Delta \left(\frac{dC_V}{dT} \right)\Bigg|_{T_{BEC}} = -3{,}66 \frac{N k_B}{T_{BEC}}$$

1930 P. Kapitza

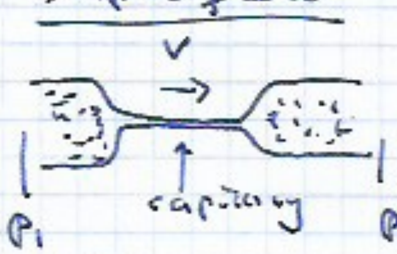
Marian Wölfke – anomaly in dielectric function

2) frictionless flow of He II

$\Updownarrow$

superflow $\qquad$ viscosity $\eta = 0$

$v < v_c$



creeping effect

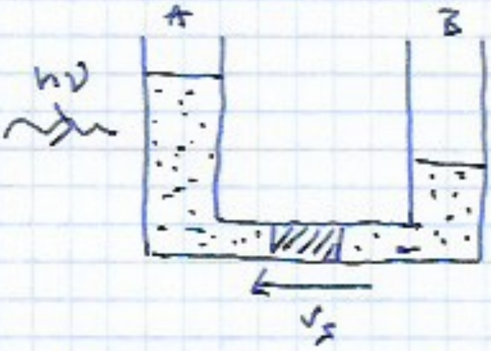
3) no boiling of the II

heat conductivity is infinite

$$\boxed{\kappa_Q = \infty} \qquad \vec{j}_Q = \kappa_Q \nabla T$$

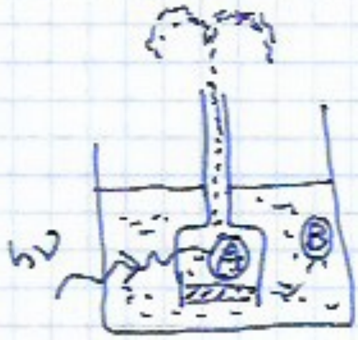
there cannot be a temperature difference

4) Thermomechanical effect

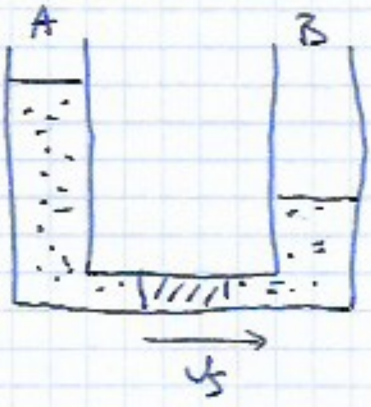


- heating A
- flow B → A
- pressure differe $\Delta p \neq 0$

fountain effect



5) Mechanothermal effect



- different levels
- flow A → B to get $\Delta p = 0$
- temperature difference

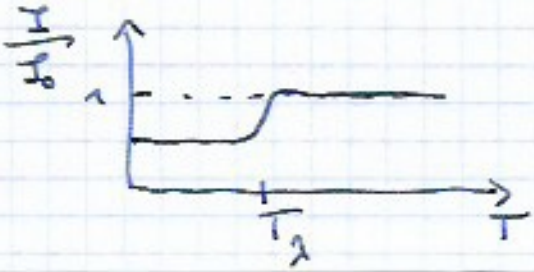
$$T_A = T + \Delta T$$
$$T_B = T - \Delta T$$

6) Andronikashvili effect

stack of disks

torsional ~~scale~~ scale

$$I \sim \omega = \frac{2\pi}{T}$$



Two fluid model - Tisza-Landau

phenomenological model describing observations for He II.

$$\begin{cases} n = n_s + n_n \\ \vec{j} = \vec{j}_s + \vec{j}_n = \vec{v}_s n_s + \vec{v}_n n_n \end{cases}$$

Superfluid component - BEC ground state

$$\psi_0(\vec{r}) = \sqrt{N_0}\,\frac{e^{i\phi}}{\sqrt{V}} = \sqrt{n_0}\,e^{i\phi}$$

excited states of superfluid component

$$\psi_s(\vec{r}) = \sqrt{n_s(\vec{r})}\,e^{i\phi(\vec{r})}$$

current density

$$\vec{j}_s(\vec{r}) = \frac{\hbar}{2mi}\left[\psi_s^*(\vec{r})\vec{\nabla}\psi_s(\vec{r}) - \psi_s(\vec{r})\vec{\nabla}\psi_s^*(\vec{r})\right] =$$

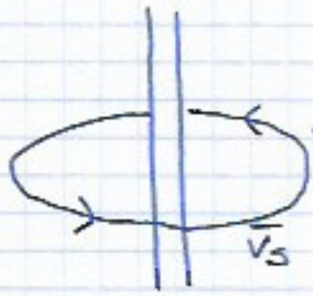
$$= \frac{\hbar}{m} n_s(\vec{r})\,\vec{\nabla}\phi(\vec{r})$$

$$\Rightarrow \boxed{\vec{v}_s(\vec{r}) = \frac{\hbar}{m}\vec{\nabla}\phi(\vec{r})}$$ velocity of the superfluid flow

Quantization of vortex

$$\vec{v}_s = \frac{\hbar}{m}\vec{\nabla}\phi(\vec{r}) \Rightarrow \boxed{\vec{\nabla}\times\vec{v}_s(\vec{r}) = 0}$$ rotationless motion of superfluid component

In a non-compact space | a space with a tube |



df. Circulation

$$\varkappa = \oint_C \bar{v}_s(\bar{r}) \cdot d\bar{l}$$

Th Does not depend on C

$$\varkappa = \frac{\hbar}{m} \oint \bar{\nabla}\phi \cdot d\bar{l} = \frac{\hbar}{m} \Delta\phi$$

For the wave function

$$e^{i\phi(0)} = e^{i\phi(L)}$$

$$\Rightarrow \quad \Delta\phi = 2\pi n \qquad n = 0, \pm 1, \pm 2, \dots$$

$$\boxed{\varkappa = \frac{h}{m} n}$$

topological (not path dependent) winding number

Two-fluid thermodynamics

- flow of superfluid component is frictionless $\eta = 0$

- entropy of superfluid component is zero

a pure quantum state $\quad S = \sum_n p_n \ln p_n = 0$

$$\text{if } p_n = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$$

Consequences

→ heat current $\vec{j}_Q = T s \vec{v}_s = 0$

entropy density

$s = 0$

$\Rightarrow \underline{\kappa_Q = \infty}$

→ thermomechanical effect

flow $\vec{v}_s$ does not equilibrate temperature differences

$$dG = -S\,dT + V\,dp + \mu \underbrace{dN}_{=0} = 0$$

$G = U - TS + pV$
Gibbs enthalpy
$G = \mu N$
$dG = 0$

$\Rightarrow p_2 - p_1 = s(T_2 - T_1)$

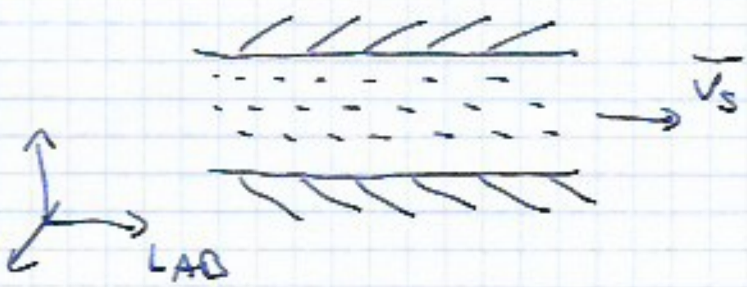
$\Rightarrow \underline{\underline{\Delta T \to \Delta p}}$

The most efficient heat engine!

Quasiparticles and superfluidity (Landau)

→ superfluid component is like a ground state with zero energy

→ when superfluid component moves with respect to laboratory frame it has finite energy



Flow of superfluid component is with zero viscosity as long as there are no quasiparticles excited, e.g. due to colisions with walls or moving body

falling body of mass M with velocity $\vec{v}$

At T=0 no excitations

If excitations are created then we have conservation laws

$$\begin{cases} \frac{1}{2} M_0 \vec{v}^2 = \frac{1}{2} M_0 \vec{v}'^2 + \varepsilon_{\vec{k}} \\ M_0 \vec{v} = M_0 \vec{v}' + \hbar \vec{k} \end{cases}$$

$\varepsilon_{\vec{k}}$, $\hbar\vec{k}$ - energy and momentum of a quasiparticle

We show under which condition the conservation laws can not be satisfied and therefore quasiparticles cannot be created $\Rightarrow$ motion must be frictionless.

$$M_0 \vec{v} - \hbar \vec{k} = M_0 \vec{v}' \quad \Big| \; {}^2$$

$$M_0^2 \vec{v}^2 - 2 M_0 \hbar \vec{v} \cdot \vec{k} + \hbar^2 \vec{k}^2 = M_0^2 \vec{v}'^2 \quad \Big| \; \frac{1}{2M_0}$$

$$\underline{\frac{1}{2} M_0 \vec{v}^2} - \hbar \vec{v} \cdot \vec{k} + \frac{1}{2M_0} \hbar^2 \vec{k}^2 = \underline{\frac{1}{2} M_0 \vec{v}'^2}$$

but $\frac{1}{2} M_0 \vec{v}^2 - \frac{1}{2} M_0 \vec{v}'^2 = \varepsilon_{\vec{k}}$

Hence $\quad \varepsilon_{\vec{k}} - \hbar \vec{v}\cdot\vec{k} + \frac{1}{2M_0}\hbar^2 k^2 = 0$

or

$$\boxed{\varepsilon_{\vec{k}} = \hbar \vec{v}\cdot\vec{k} - \frac{1}{2M_0}\hbar^2 k^2}$$

There is a minimal $\vec{v}$ when this equation is satisfied. Then $\vec{v} \parallel \vec{k}$ and

$$v_c = \min\left\{ \frac{\varepsilon_{\vec{k}} + \frac{1}{2M_0}\hbar^2 k^2}{\hbar k} \right\}$$

When $M_0 \to \infty$ we get $\boxed{v_c = \min\left(\frac{\varepsilon_{\vec{k}}}{\hbar k}\right)}$

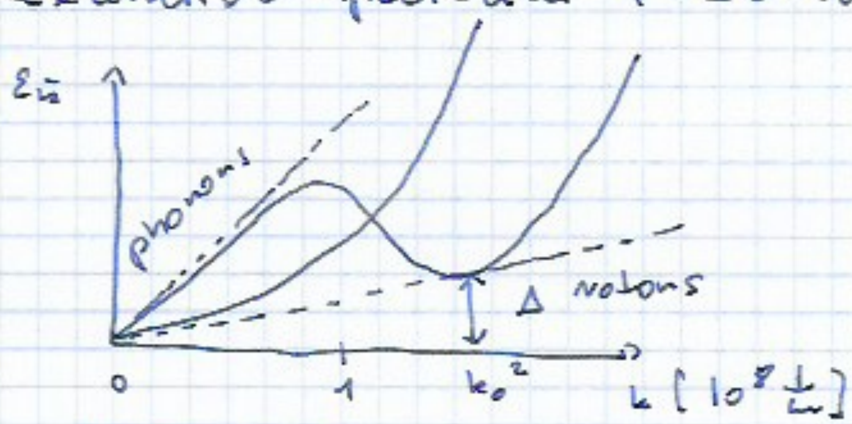
•) free atoms

$$v_c = \min\left(\frac{\frac{\hbar^2 k^2}{2m}}{\hbar k}\right) = 0$$

•) phonons $\quad \varepsilon_k = \hbar v_s k$

$$v_c = \min\left(\frac{\hbar v_s k}{\hbar k}\right) = v_s > 0 \quad !$$

Real excitation spectrum (neutron scattering)



$$\boxed{v_c = \frac{\Delta}{\hbar k_0} \simeq 5\cdot 10^3 \ \mathrm{cm/s}}$$

critical velocity in the II.