IV PLANCK DISTRIBUTION APPLICATIONS

§ 1. PLANCK DISTRIBUTION

A universal Planck distribution describes a radiation of electromagnetic field from a resonant cavity, thermal fluctuations of electric current, spectrum of ion oscillations in crystals.

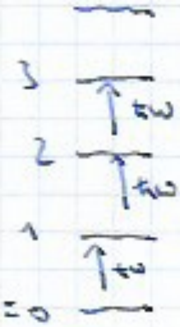
Consider a single mode of vibration of frequency $\boxed{f = \frac{1}{T}}$ [1 Hz = 1/s] (częstotliwość).
We also use $\boxed{\omega = 2\pi f}$ $\left[\frac{rad}{s}\right]$ (częstość).
Vibrations can be mechanical or electrical.

Planck, 1900 – a single mode of vibration with frequency ω is created in discrete portion of energies, such that energy of system is

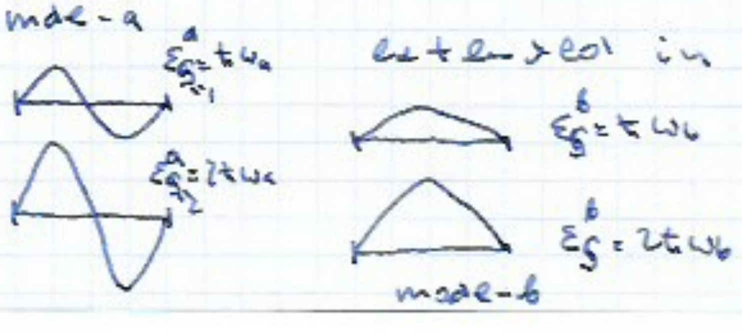
$$\varepsilon_s = \hbar\omega\left(s+\tfrac{1}{2}\right) = hf\left(s+\tfrac{1}{2}\right)$$

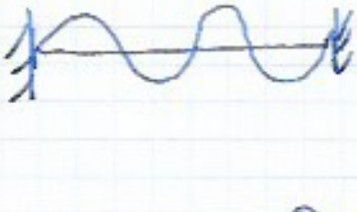
$s = 0, 1, 2, 3, \dots$ ← zero mode energy

↑ spectrum of harmonic oscillator



- harmonic oscillator was localized
- electromagnetic oscillations or mechanical oscillations are extended in space





s → quantum number of an oscillator of frequency ω

→ number of given mode of frequency ω

<u>canonical ensemble</u>

$$\boxed{p(s) = \frac{1}{Z} e^{-\beta \varepsilon_s} = \frac{1}{Z} e^{-\beta \hbar\omega (s+\frac{1}{2})}}$$

<u>Partition function</u>

$$\boxed{Z = \sum_{s=0}^{\infty} \left(e^{-\beta\hbar\omega}\right)^s e^{-\beta\frac{\hbar\omega}{2}} = \frac{e^{-\beta\frac{\hbar\omega}{2}}}{1-e^{-\beta\hbar\omega}}}$$

$\sum_{s=0}^{\infty} x^s = \frac{1}{1-x}$, $|x|<1$ geometric progress, $e^{-\beta\hbar\omega}<1$

$\underline{\omega>0}$, $\underline{\beta>0}$

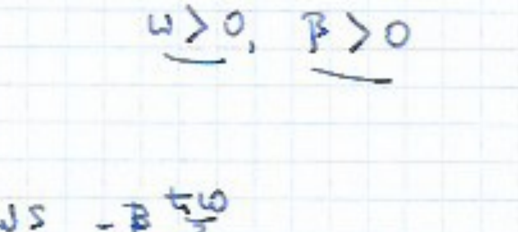
<u>Average number of modes</u>

$$\langle s\rangle = \sum_{s=0}^{\infty} s\, p(s) = \frac{1}{Z}\sum_{s=0}^{\infty} s e^{-\beta\hbar\omega s} e^{-\beta\frac{\hbar\omega}{2}}$$

note:
$$\sum_s s e^{-ys} = -\frac{d}{dy}\sum_s e^{-ys} = -\frac{d}{dy}\left(\frac{1}{1-e^{-y}}\right) = \frac{e^{-y}}{(1-e^{-y})^2}$$
$y = \hbar\omega\beta$

$$\langle s\rangle = \frac{1}{\frac{e^{-\beta\hbar\omega/2}}{1-e^{-\beta\hbar\omega}}} \frac{e^{-\hbar\omega\beta}}{(1-e^{-\beta\hbar\omega})^2} e^{-\beta\hbar\omega/2} = \frac{e^{-\beta\hbar\omega}}{1-e^{-\beta\hbar\omega}}$$

$$\boxed{\langle s\rangle = \bar{s} = \frac{1}{e^{\beta\hbar\omega}-1}}$$



<u>Planck distribution</u> for a simple mode of vibration in thermal equilibrium with environment.

Internal energy of a single mode

$$\boxed{u = \langle \varepsilon \rangle = \hbar\omega\left(\langle n \rangle + \tfrac{1}{2}\right) = \frac{\hbar\omega}{e^{\beta\hbar\omega}-1} + \frac{\hbar\omega}{2}}$$

•) low temperature limit $k_B T \ll \hbar\omega$ (quantum)

$$Z \simeq e^{-\beta\frac{\hbar\omega}{2}} \qquad\qquad e^{\frac{\hbar\omega}{k_BT}} \gg 1$$

$$u \simeq \hbar\omega\, e^{-\frac{\hbar\omega}{k_BT}} + \text{const.}$$

•) high temperature limit $k_BT \gg \hbar\omega$ (classical)

$$Z \simeq \frac{k_BT}{\hbar\omega}$$

$$u \simeq \frac{\hbar\omega}{1+\frac{\hbar\omega}{k_BT}+\dots-1} + \text{const.} = k_BT + \text{const.}$$

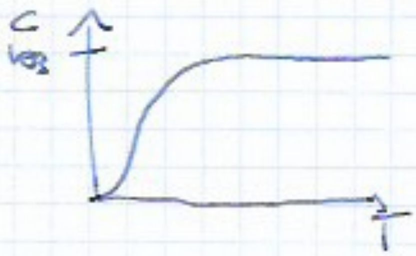
Warning: popular formulation of equipartition theorem: "on each degree of freedom there is $\frac{1}{2}k_BT$ energy" is not correct

•) Specific heat

$$\boxed{C = \frac{du}{dT} = \frac{(\hbar\omega)^2}{4(k_BT)^2\sinh^2\left(\frac{\hbar\omega}{2k_BT}\right)}}$$

- no zero modes

$$C = \begin{cases} k_B & T \to \infty \\ \frac{(\hbar\omega)^2}{(k_BT)^2}\, e^{-\frac{\hbar\omega}{k_BT}} & T \to 0 \end{cases}$$



•) Free energy

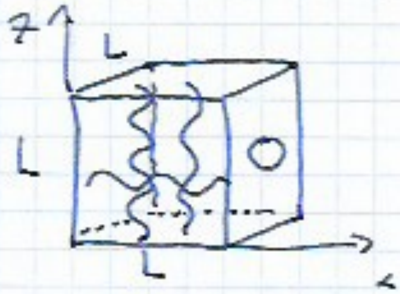
$$\boxed{F = -k_BT \ln Z = \frac{\hbar\omega}{2} + k_BT \ln\left(1-e^{-\beta\hbar\omega}\right)}$$

•) entropy $\quad F = u - TS \Rightarrow S = \frac{u-F}{T}$

$$\boxed{S = \frac{1}{T}\left(\frac{\hbar\omega}{e^{\beta\hbar\omega}-1} - k_BT \ln\left(1-e^{-\beta\hbar\omega}\right)\right)}$$ - no zero modes

§2. BLACK BODY RADIATION

- resonance cavity – a metallic box with a hole, there are infinitely many electromagnetic modes $^{(n)}$ with energies $\hbar\omega_n$



Walls are ideal conductors with $\overline{E} = 0$

Boundary conditions

at Σ $\quad \overline{E}_{\parallel} = 0 \quad (\overline{B}_{\perp} = 0)$

For a cube of length L: components of electric field

$$E_x = E_{x0} \sin\omega t \cos\left(n_x \frac{\pi x}{L}\right) \sin\left(n_y \frac{\pi y}{L}\right) \sin\left(n_z \frac{\pi z}{L}\right)$$

$$E_y = E_{y0} \sin\omega t \sin\left(n_x \frac{\pi x}{L}\right) \cos\left(n_y \frac{\pi y}{L}\right) \sin\left(n_z \frac{\pi z}{L}\right)$$

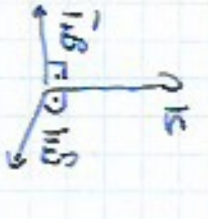
$$E_z = E_{z0} \sin\omega t \sin\left(n_x \frac{\pi x}{L}\right) \sin\left(n_y \frac{\pi y}{L}\right) \cos\left(n_z \frac{\pi z}{L}\right)$$

tangent components vanish on Σ

Maxwell equations: $\quad \overline{\nabla} \cdot \overline{E} = 0$

$$0 = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = E_{x0} n_x + E_{y0} n_y + E_{z0} n_z = \overline{E_0} \cdot \overline{n}$$

$\overline{E} \perp \overline{n}$, transverse polarization. For given $\overline{n}$ two polarization possible



next Maxwell equations:

$$\frac{1}{c^2} = \varepsilon_0 \mu_0 \begin{cases} \overline{\nabla} \times \overline{E} = -\dfrac{\partial \overline{B}}{\partial t} \quad / \overline{\nabla} \times \\ \overline{\nabla} \times \overline{B} = \dfrac{1}{c^2} \dfrac{\partial \overline{E}}{\partial t} \end{cases}$$

$$\underbrace{\overline{\nabla} \times \overline{\nabla} \times \overline{E}}_{\overline{\nabla}\, \overline{\nabla} \cdot \overline{E} - \Delta \overline{E}} = -\frac{\partial}{\partial t} \overline{\nabla} \times \overline{B} = -\frac{1}{c^2} \frac{\partial^2 \overline{E}}{\partial t^2}$$

but $\overline{\nabla} \cdot \overline{E} = 0$

We get wave equation $\quad \boxed{\Delta \overline{E} = \frac{1}{c^2} \frac{\partial^2 \overline{E}}{\partial t^2}}$

using our solutions, e.g.

$$c^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)E_z(x,y,z,t)=\frac{\partial^2 E_z(x,y,z,t)}{\partial t^2}$$

$$\Rightarrow c^2\pi^2(n_x^2+n_y^2+n_z^2)=\omega^2 L^2$$

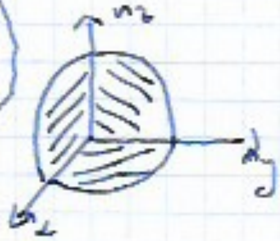
Hence we obtain a dispersion

An energy of a single mode n $\hbar\omega_n$

$$\boxed{\omega_n=\frac{\pi c}{L}n,\qquad n=\sqrt{n_x^2+n_y^2+n_z^2}}$$

Energy of many modes

discrete quanta of energy - photons



$$U=2\sum_{\vec n}\langle\varepsilon_n\rangle=2\sum_{\vec n}\frac{\hbar\omega_n}{e^{\beta\hbar\omega_n}-1}$$

$$\sum_{\vec n}=\sum_{n_x n_y n_z}\xrightarrow[L\to\infty]{}\frac{1}{8}\int_0^\infty 4\pi n^2\,dn$$

$$U=2\cdot\frac{1}{8}\cdot4\pi\int_0^\infty dn\, n^2\frac{\hbar\omega_n}{e^{\beta\hbar\omega_n}-1}=\frac{\pi^2\hbar c}{L}\int_0^\infty dn\frac{n^3}{e^{\frac{\hbar\pi c}{k_B T L}n}-1}$$

$$=\left\{x=\frac{\hbar c\pi}{k_B T L}n\right\}=\frac{\pi^2\hbar c}{L}\left(\frac{k_B T L}{\hbar c\pi}\right)^4\underbrace{\int_0^\infty dx\frac{x^3}{e^x-1}}_{\frac{\pi^4}{15}}$$

energy density $\qquad V=L^3$

$$\boxed{u=\frac{U}{V}=\frac{\pi^2}{15\hbar^3c^3}(k_BT)^4}$$

Stefan-Boltzmann Law

radiation energy density is proportional to T^4

Stefan-Boltzman constant

$$\sigma=\frac{\pi^2k_B^4}{60\hbar^3c^2}=5.67\cdot10^{-8}\frac{J}{s\,m^2K^4}$$

$$\boxed{u=\frac{4\sigma}{c}T^4}$$

Thermodynamics of EM radiation (tutorial)

$$Z = \prod_n \frac{e^{-\beta\hbar\omega_n/2}}{1-e^{-\beta\hbar\omega_n}} = e^{-\beta F}$$ partition function

$$F = -\frac{4\sigma}{3c} V T^4 = -k_B T \ln Z$$ free energy

$$S = -\left(\frac{\partial F}{\partial T}\right)_V = \frac{16\sigma}{3c} V T^3$$ entropy

$$p = -\left(\frac{\partial F}{\partial V}\right)_T = \frac{4\sigma}{3c} T^4$$ pressure

$$c_V = T\left(\frac{\partial S}{\partial T}\right)_V = \left(\frac{\partial U}{\partial T}\right)_V = \frac{16\sigma}{c} V T^3$$ - specific heat

Equation of states
Stefan-Boltzmann law $\boxed{U = 3pV}$ not $U = \frac{3}{2}pV$!

e.g. $T = 10^5$ K (nuclear explosion) $p = 0{,}25$ bar

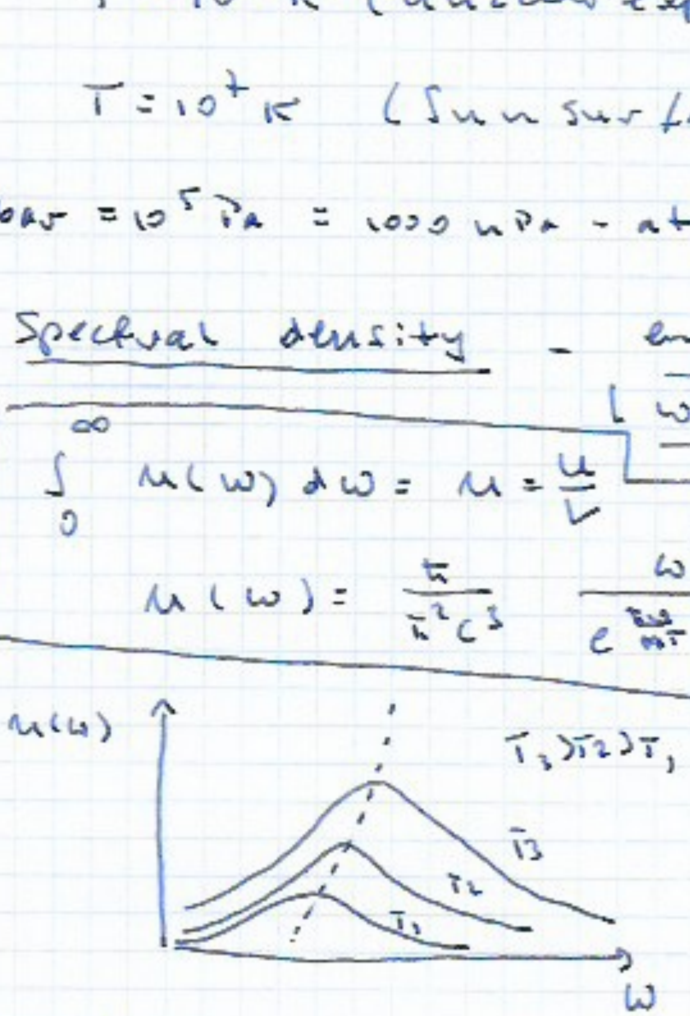
$T = 10^7$ K (Sun surface) $p = 25\cdot 10^6$ bar

[1 bar $= 10^5$ Pa $= 1000$ hPa - atmospheric pressure on sea level]

Def. Spectral density - energy per volume in $[\omega, \omega + d\omega]$ range $u(\omega)$

$$\boxed{\int_0^\infty u(\omega)\, d\omega = u = \frac{U}{V} \qquad u(\omega) = \frac{\hbar}{\pi^2 c^3}\frac{\omega^3}{e^{\frac{\hbar\omega}{k_BT}}-1}}$$

(1900)
Planck law



$$x_{max} = \frac{\hbar\omega_{max}}{k_B T}$$

$\rightsquigarrow$ $\hbar\omega_{max} = 2{,}82\, k_B T$

one can estimate T from ω_{max} (e.g. a star surface)

$$\hbar\omega = \frac{hc}{\lambda}$$

Law of Wien's shift

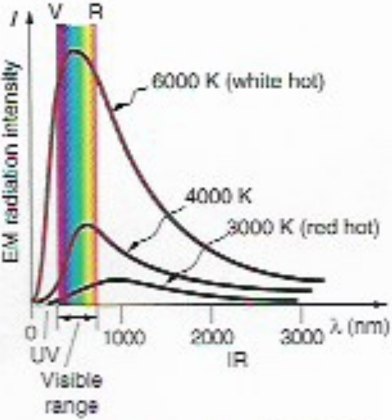
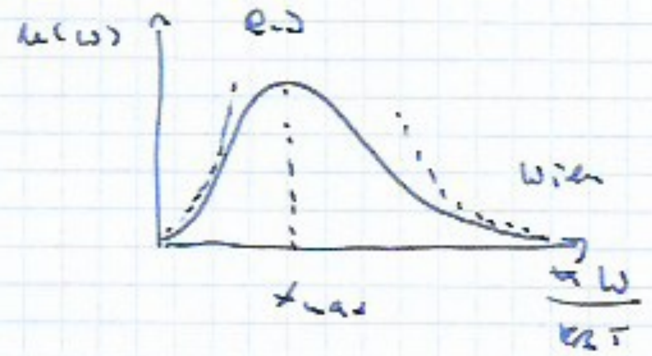


Figure 27.2 Graphs of blackbody radiation (from an ideal radiator) at three different temperatures. The intensity or rate of radiation emission increases dramatically with temperature, and the peak of the spectrum shifts toward the visible and ultraviolet parts of the spectrum. The shape of the spectrum cannot be described with classical physics.



- <u>classical limit</u> $\hbar\omega \ll k_B T$

$$u(\omega) \simeq \frac{\hbar}{\pi^2 c^3} \frac{\omega^3 k_B T}{\hbar\omega} = \frac{k_B T \omega^2}{\pi^2 c^3}$$ Rayleigh-Jeans law

other derivation: equipartition theorem $\langle E \rangle = k_B T$

energy density $$u(\omega, T) = \frac{(\text{energy in } [\omega, \omega + d\omega]) \cdot (k_B T)}{\text{volume}}$$

Density of states; average energy

$$\left(2 \cdot \frac{1}{8} \cdot 4\pi \int_0^\infty dn\, n^2 = \left\{ \omega = \frac{\pi c}{L} n \right\} = \frac{L^3}{\pi^2 c^3} \int_0^\infty \omega^2 d\omega \right)$$

Density of states $\rho(\omega) = \frac{V}{\pi^2 c^3} \omega^2$

$$u(\omega, T) = \frac{1}{V} \rho(\omega) k_B T = \frac{\omega^2}{\pi^2 c^3} k_B T$$

UV cathastrophe

$$\int_0^\infty d\omega\, u(\omega, T) \sim \int_0^\infty d\omega\, \omega^2 = \infty$$

- <u>UV limit</u> $\hbar\omega \gg k_B T$

$$u(\omega, T) \simeq \frac{\hbar\omega^3}{\pi^2 c^3} e^{-\frac{\hbar\omega}{k_B T}}$$ Wien's law

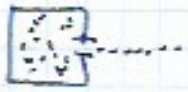
<u>Average number of excited modes</u>

$$N = 2 \sum_n \frac{1}{e^{\beta\hbar\omega_n} - 1} = \dots = \frac{V}{\pi^2 c^3} \int_0^\infty d\omega \frac{\omega^2}{e^{\beta\omega\hbar} - 1} =$$

$$= \frac{V (k_B T)^3}{\pi^2 c^3 \hbar^3} \int_0^\infty dx \frac{x^2}{e^x - 1} = \frac{2\zeta(3)}{\pi^2} V \left(\frac{k_B T}{\hbar c}\right)^3$$

$$\boxed{N \simeq 0{,}244 \cdot V \cdot \left(\frac{k_B T}{\hbar c}\right)^3 \xrightarrow[T \to 0]{} 0}$$

Radiation from a cavity – black body radiation



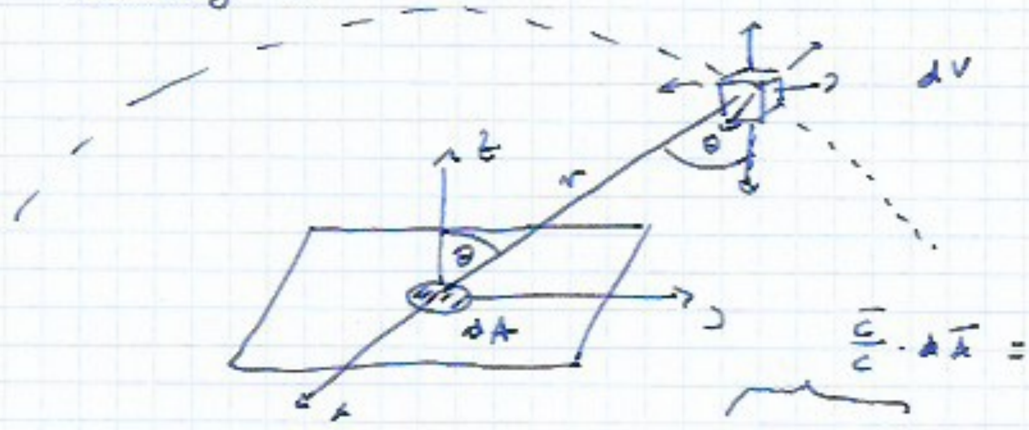
·) distribution of radiation – isotropic

··) photons move with velocity c

→ amount of radiation energy in $[\omega, \omega+d\omega]$ and in volume $dV = r^2 dr \sin\theta d\theta d\varphi$ is given by

$$u(\omega)\, d\omega\, dV$$

→ radiation travels from dV in all direction with velocity c



$$\underbrace{\frac{\vec{c}}{c}\cdot d\vec{A}}_{dA\cos\theta} = \vec{n}\cdot dA \cdot \frac{\vec{c}}{c}$$

→ a part of radiation $\frac{dA\cos\theta}{4\pi r^2}$ ← sphere area

ratio $\frac{dA}{4\pi r^2}$

is directed onto a surface element dA

→ the amount of radiation in $[\omega, \omega+d\omega]$, which goes through dA in time Δt, is a sum from each contributions from a hemisphere of radius $c\Delta t$

$$\Delta E = \int_0^{c\Delta t} dr \underbrace{\int_0^{\pi/2} d\theta \int_0^{2\pi} d\phi}_{\text{hemisphere}}\, r^2 \sin\theta\, u(\omega)\, d\omega\, \frac{dA\cos\theta}{4\pi r^2} =$$

$$= \frac{1}{2}\, dA\, d\omega\, u(\omega) \underbrace{\int_0^{c\Delta t} dr \int_0^{\pi/2} \sin\theta\cos\theta\, d\theta}_{\frac{1}{2} c\, dt}$$

$$\boxed{\Delta E = \frac{c}{4} u(\omega)\, d\omega\, dt\, dA} \Rightarrow \boxed{J(\omega,T) = \frac{c}{4} u(\omega,T)}$$

energy flux – an energy per area, per time, in $d\omega$

Total energy flux

$$\boxed{J(T) = \int_0^\infty d\omega\, J(\omega,T)\, d\omega = \sigma T^4}$$

Stefan – Boltzmann law

Photometric quantities

Df. **Luminous intensity** (natężenie światła) (sun, moon, etc.)

$$\boxed{I = \frac{\Delta W}{\Delta t \, \Delta\Omega}}$$

energy per time and per solid angle

$$[I] = \left[\frac{J}{s \cdot sr}\right]$$

$$I = I(\theta, \varphi, T)$$

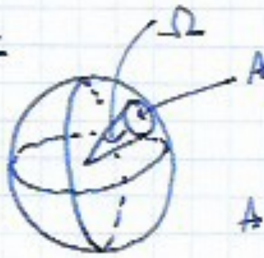
radian

$\varphi = \frac{l}{r}$ [rad]

$l = 2\pi r$

$\varphi = 2\pi$ [rad]

steradian



$\Omega = \frac{A}{r^2}$ [sr]

$A = 4\pi r^2$

$\Omega = 4\pi$ [sr]

Df. SI - candela [cd]

Intensity of a source, which radiates monochromatic wave with $\lambda = 555$ [nm] and the power $\frac{1}{683}$ [W] in a solid angle of 1 [sr].

$\lambda = 555$ nm $\Leftrightarrow f = 540 \cdot 10^{12}$ Hz

$\frac{1}{683}$ W $\Leftrightarrow$ 1,47 mW

candela - candela

lumen - light

lux - luminosity

Df **Luminous flux** (strumień światła) total power of a radiation, going through a solid angle Ω_0.

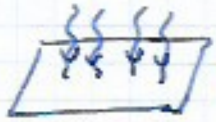
$$\boxed{\Phi = \int_{\Omega_0} I(\theta, \varphi; T)\, d\Omega} \Leftrightarrow \boxed{I(\theta, \varphi; T) = \frac{d\Phi}{d\Omega}}$$

$[\Phi] = [1\ lm = 1\ cd \cdot 1\ sr]$ - lumen (lm)

Df **Illuminance** (natężenie oświetlenia)

$$\boxed{E = \frac{\Phi}{A}}$$

flux of light per area in perpendicular direction

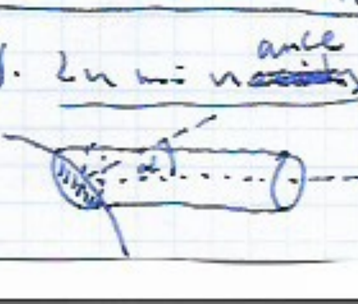


$[E] = [1\ lx = \frac{1\ lm}{1\ m^2}]$ - [lux]

Df. Lumin~~osity~~ance

$$L = \frac{\Delta I}{\Delta A \cos\alpha} = L(\theta, \varphi)$$

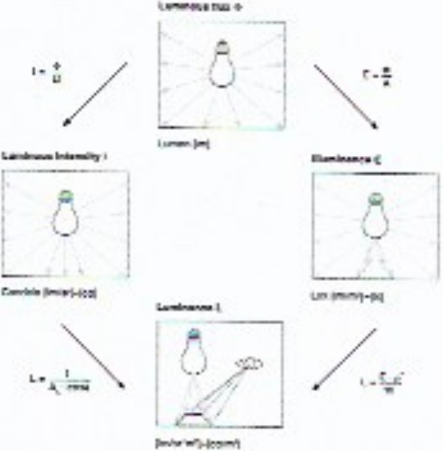
- intensity from the area ΔA per projection of ΔA



$\Delta\Omega$

$[L] = [\frac{1\ cd}{1\ m^2} = 1\ nt]$ (nit)

Luminous flux – Luminous intensity – Illuminance – Luminance



Ω = solid angle into which luminous flux is emitted
A = area hit by luminous flux
$A_L \cdot \cos\varepsilon$ = visible areas of light sources
ρ = reflectance of area
π = 3.14
* = for diffuse surface areas

E.g. Illuminance [lx]

room inside	weak	20 - 40	Weber law
	medium	40 - 80	
	strong	80 - 100	(feeling) ≈
screen in cinema		80 - 250	log (source)
paper in sunshine		100 000	

E.g. Electric bulb

E14

7 W

470 lm $\Leftrightarrow$ ◯ in 4π sr

warm white

25 000 h

Absorption, Emission, Kirchhoff's law

Df. absorption

$$\boxed{A(\omega, T) = \frac{\text{absorbed energy at } \omega}{\text{total energy at } \omega}}$$

$[A] = [1]$

$A = A(\omega, T, \text{chemistry, colour, phase, etc})$

$A(\omega,T) = \text{const} < 1 \quad \forall \omega$ – gray body

$A(\omega,T) = 1$ – black body

$A(\omega,T) = 0$ – perfect mirror

Df. emission

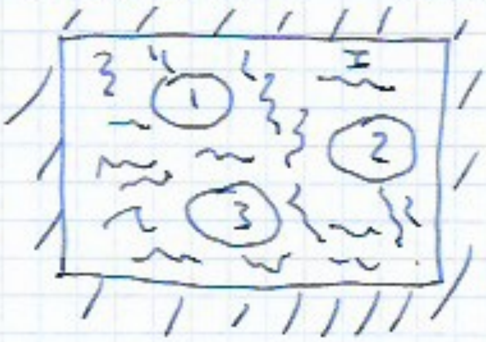
$E(\omega, T) =$ emitted energy at ω per time

$[E] = [W = \frac{J}{s}]$

1859 – <u>Kirchhoff's Law</u>

$$\boxed{\frac{E(\omega,T)}{A(\omega,T)} = \varepsilon(\omega,T) \text{ – universal function of } \omega \text{ and } T \text{ for all bodies.}}$$

<u>Proof</u> N – bodies in thermal equilibrium with radiation inside an isolated mirror box.



(*) Each body emits and absorbs the same amount of energy by ΔS and Δt

$$E_1(\omega,T)\,\Delta S_1\,\Delta t = A_1(\omega,T)\,\Delta S_1\,\Delta t\; I(\omega,T)$$

I light intensity (the same for all bodies)

$$E_2(\omega,T)\,\Delta S_2\,\Delta t = A_2(\omega,T)\,\Delta S_2\,\Delta t\, I(\omega,t)$$

$\vdots$

$$\Rightarrow \frac{E_1(\omega,T)}{A_1(\omega,T)} = \frac{E_2(\omega,T)}{A_2(\omega,T)} = \dots = I(\omega,T) \equiv \varepsilon(\omega,T)$$

□

For a black body ~~radiation~~ $A_{bb}(\omega,T) = 1$

$$\frac{E_{bb}(\omega,T)}{A_{bb}(\omega,T)} = \varepsilon(\omega,T) = I(\omega,T) = \frac{c}{4}u(\omega,T)$$

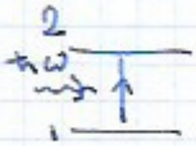
<u>Planck distribution</u>

$$\Rightarrow \boxed{\varepsilon(\omega,T) = \frac{c}{4}u(\omega,T) \text{ – Planck distribution for all bodies.}}$$

(*) A body more radiates if it more absorps.
E.g. black copper cools faster than water at the same T.

Einstein coefficients

1) **absorption**



transition rate

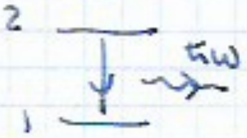
$$\left(\frac{dN}{dt}\right)_{ab} = N_1 \, B_{12} \, u(\omega)$$

occupation of initial states

proportionality coefficient

density energy

2) **spontaneous emission**

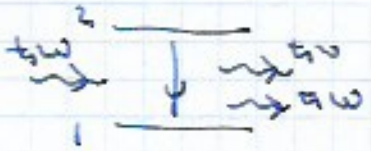


$$\left(\frac{dN}{dt}\right)_{sp-em} = A \, N_2$$

coefficient

occupation of initial state

3) **stimulated emission**



$$\left(\frac{dN}{dt}\right)_{st-em} = N_2 \, B_{21} \, u(\omega)$$

Detailed balance condition

$$\left(\frac{dN}{dt}\right)_{ab} = \left(\frac{dN}{dt}\right)_{sp\,em} + \left(\frac{dN}{dt}\right)_{st-em}$$

$$N_1 B_{12} u(\omega) = N_2 A + N_2 B_{21} u(\omega)$$

In thermal equilibrium

$$N_1 = \text{const}\; e^{-\beta E_1}, \quad N_2 = \text{cost.}\; e^{-\beta E_2} \Rightarrow \frac{N_1}{N_2} = e^{-\beta(E_1-E_2)} = e^{\beta\hbar\omega}$$

$$\rightarrow u(\omega) = \frac{N_2 A}{N_1 B_{12} - N_2 B_{21}} = \frac{A}{B_{21}} \frac{1}{\frac{B_{12}}{B_{21}} e^{\beta\hbar\omega} - 1}$$

In the limit $T \to \infty$

$$B_{12} = B_{21} = B \Rightarrow u(\omega) = \frac{A}{B} \frac{1}{e^{\beta\hbar\omega} - 1}$$

absorption = stimulated emission

comparing with Planck distribution

$$\boxed{\frac{A}{B} = \frac{\omega^3 \hbar}{\pi^2 c^3}}$$

A and B are Einstein coefficients

Relation to absorption and emission

$$\boxed{\frac{E(\omega,T)}{A(\omega,T)} = \frac{c}{4}\, u(\omega,T) = \frac{c}{4}\frac{A}{B}\frac{1}{e^{\beta\hbar\omega}-1}}$$

Dimensional analysis

$[N_2] = 1$

$[A]$: $\left[\frac{1}{s}\right] = \left[\left(\frac{dN}{dt}\right)_{sp-em}\right] = A N_2 \Rightarrow [A] = \left[\frac{1}{s}\right]$

$[B]$: $\left[\frac{1}{s}\right] = \left[\left(\frac{dN}{dt}\right)_{st-em}\right] = \left[B\, N_2\, u(\omega)\right] =$

$$= \left[B\, N_2\, \underbrace{\frac{\hbar\omega^3}{\pi^2 c^3}}_{Js}\, \frac{1}{e^{\beta\hbar\omega}-1}\right]$$

$[B] = \left[\frac{1}{s} \cdot \frac{\omega^3/s^3}{J\cdot s \cdot 1/s^3} = \frac{\omega^3}{s}\frac{1}{kg\cdot \frac{m^2}{s^2}\cdot s} = \frac{m}{kg}\right]$

$$\Rightarrow \boxed{\left[\frac{A}{B}\right] = \left[\frac{kg}{m\cdot s}\right]}$$

On the other hand:

$$\frac{E(\omega,T)}{A(\omega,T)} = \frac{A}{B}\frac{c}{4}\frac{1}{e^{\beta\hbar\omega}-1}$$

$\left[\frac{E}{A}\right] = \left[\frac{J/m^2}{1}\right] = \left[\frac{kg\cdot \frac{m^2}{s^2}}{m^2} = \frac{kg}{s^2}\right]$

$$\Rightarrow \boxed{\left[\frac{A}{B}\right] = \left[\frac{kg}{s^2}\frac{1}{m/s}\right] = \left[\frac{kg}{m\cdot s}\right]}$$

$\square$