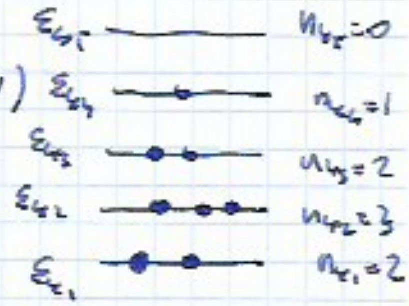
§2. FERMI-DIRAC AND BOSE-EINSTEIN DISTRIBUTION FUNCTIONS

Grand partition function



$$\Xi = \sum_{N=0}^{\infty} \sum_{E_\xi(N)} e^{-\beta\left(\sum_{\xi\sigma} \varepsilon_\xi n_\xi - \mu N\right)}$$

Since $E_{(N)} = \sum_{\xi\sigma} \varepsilon_\xi n_\xi$ for non-interacting particles.

ε_ξ: orbital energy

n_ξ: number of particles in a given orbital (indistinguishability)

$$\Xi = \sum_{N=0}^{\infty} \sum_{n_\xi}^{\left(\sum_{\xi\sigma} n_\xi = N\right)} e^{-\beta\left(\sum_{\xi\sigma} \varepsilon_\xi n_\xi - \mu \sum_{\xi\sigma} n_\xi\right)} =$$

$$= \sum_{N=0}^{\infty} \prod_{\xi\sigma} \sum_{n_\xi}^{\left(\sum_\xi n_\xi = N\right)} e^{-\beta(\varepsilon_\xi - \mu) n_\xi} =$$

$$= \prod_{\xi\sigma} \sum_{n_\xi}^{\infty} \left(e^{-\beta(\varepsilon_\xi - \mu)}\right)^{n_\xi}$$

$$\sigma = \underbrace{-s, -s+1, \dots, s-1, s}_{\rho = 2s+1}$$

degeneracy

$$n_\xi = \begin{cases} 0, 1 & \text{fermions} \\ 0, 1, 2, \dots \infty & \text{bosons} \end{cases}$$

$$\boxed{\begin{aligned} \Xi_f &= \prod_{\xi\sigma}\left(1 + e^{-\beta(\varepsilon_\xi - \mu)}\right) \\ \Xi_b &= \prod_{\xi\sigma}\left(1 + e^{-\beta(\varepsilon_\xi - \mu)} + \left(e^{-\beta(\varepsilon_\xi - \mu)}\right)^2 + \dots\right) = \prod_{\xi}\left(\frac{1}{1 - e^{-\beta(\varepsilon_\xi - \mu)}}\right) \end{aligned}}$$

converges if

$\forall \xi \quad e^{\beta(\varepsilon_\xi - \mu)} < 1$

Taking into account g - degeneracy factor

$$\boxed{\Xi_{f,b}(T,V,\mu) = \prod_{\vec{k}} \left(1 \pm e^{-\beta(\varepsilon_{\vec{k}}-\mu)}\right)^{\pm g}}$$

(+) - fermions

(−) - bosons

Grand canonical potential

$$\boxed{\Phi_{f,b}(T,V,\mu) = -k_B T \ln \Xi_{f,b} = \underset{b}{\overset{f}{\mp}} k_B T g \sum_{\vec{k}} \ln\left(1 \underset{b}{\overset{f}{\pm}} e^{-\beta(\varepsilon_{\vec{k}}-\mu)}\right)}$$

Average number of particles

$$\bar{N} = -\left(\frac{\partial \Phi}{\partial \mu}\right)_{T,V} \underset{\text{(tutorials)}}{=} g \sum_{\vec{k}} \frac{1}{e^{\beta(\varepsilon_{\vec{k}}-\mu)} \pm 1} = \sum_{\vec{k}} \langle n_{\vec{k}} \rangle$$

Orbital occupation

$$\langle n_{\vec{k}} \rangle = \frac{1}{\Xi} g \sum_{n_{\vec{k}}} e^{-\beta(\varepsilon_{\vec{k}}-\mu)n_{\vec{k}}} n_{\vec{k}} \overset{\text{(tutorials)}}{=}$$

$$= g \frac{1}{e^{\beta(\varepsilon_{\vec{k}}-\mu)} \pm 1}$$

must be <u>non-negative</u>

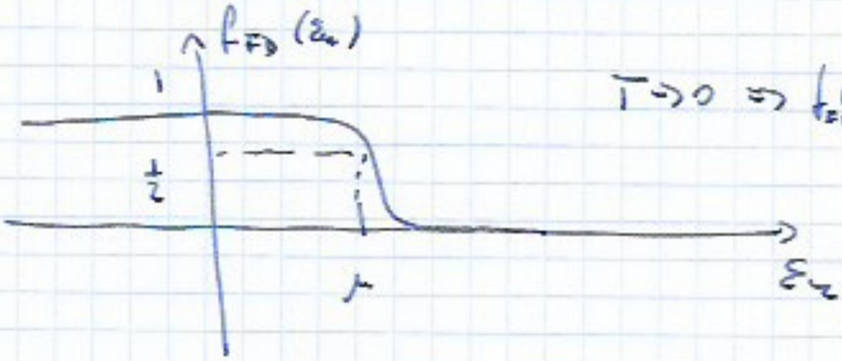
Fermi-Dirac and Bose-Einstein functions

$$\boxed{f_{\substack{FD\\BE}}(\varepsilon_{\vec{k}}) = \frac{1}{e^{\beta(\varepsilon_{\vec{k}}-\mu)} \pm 1}}$$

(+) FD

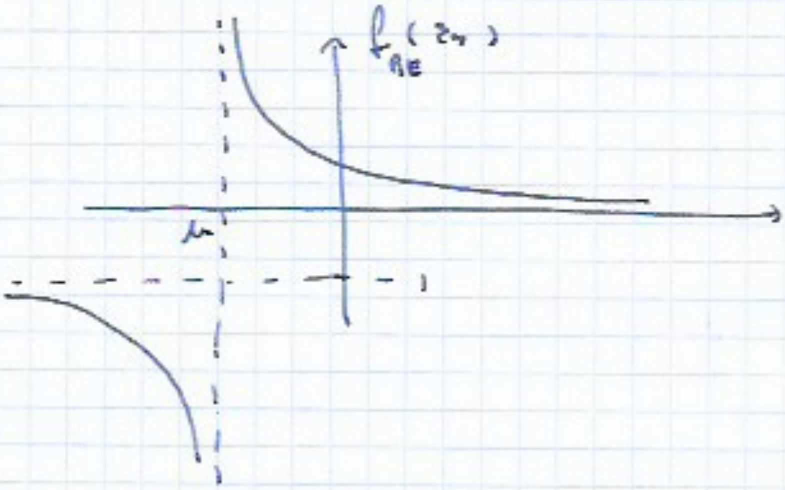
(−) BE

FD



$T \to 0 \Rightarrow f_{FD}(\varepsilon_k) = \Theta(\mu - \varepsilon_k)$

BE



only for $\varepsilon_{\vec{k}} > \mu$

$f(\varepsilon_{\vec{k}}) > 0$

geometric series converges if $e^{\beta(\varepsilon_{\vec{k}}-\mu)} < 1$.

§ 3. QUANTUM CORRECTIONS TO IDEAL GAS

Classical limit $n \ll n_Q$

$$\mu = k_B T \ln\left(\frac{n}{n_Q}\right) \xrightarrow[n \ll n_Q]{} -\infty$$

and $T \to \infty$ $(\beta \to 0)$

What are quantum corrections to the ideal gas?

$$\boxed{\Phi_{f}(T, V, \mu) = \mp k_B T g \sum_{\vec{k}} \ln\left(1 \pm e^{-\beta(\varepsilon_{\vec{k}} - \mu)}\right)}$$

$\vec{k} = \frac{2\pi}{L}(n_x, n_y, n_z)$, $\Delta n_i = 1$, $\Delta k_i = \left(\frac{L}{2\pi}\right)^{-1} \Delta n_i$

$$\sum_{\vec{k}} = \sum_{n_x} \sum_{n_y} \sum_{n_z} = \sum_{n_x} \Delta n_x \sum_{n_y} \Delta n_y \sum_{n_z} \Delta n_z =$$

$$= \left(\frac{L}{2\pi}\right)^3 \sum_{k_x} \Delta k_x \sum_{k_y} \Delta k_y \sum_{k_z} \Delta k_z \xrightarrow[L \to \infty]{} \left(\frac{L}{2\pi}\right)^3 \int dk_x \int dk_y \int dk_z$$

$$e^{-\beta(\varepsilon_{\vec{k}} - \mu)} \ll 1$$

expand

$$\ln(1 \pm x) = \pm x - \frac{1}{2} x^2$$

$$\Phi_{f}(T, V, \mu) = \mp g k_B T \frac{V}{8\pi^3} \int d^3k \left[\pm e^{-\beta(\varepsilon_{\vec{k}} - \mu)} - \frac{1}{2} e^{-2\beta(\varepsilon_{\vec{k}} - \mu)} \right] =$$

$$= -g k_B T \frac{V}{8\pi^3} \underbrace{e^{\beta\mu}}_{\alpha} \int d^3k \, e^{-\beta \frac{\hbar^2 k^2}{2m}} \pm \frac{1}{2} g k_B T \frac{V}{8\pi^3} \underbrace{\left(e^{\beta\mu}\right)^2}_{\alpha^2} \int d^3k \, e^{-\beta \frac{\hbar^2 k^2}{m}} + \dots$$

activity

$\alpha = e^{\beta\mu} \ll 1 \to$ a formal expansion in the activity, virial expansion

$$\boxed{\Phi_{f,b}(T,V,\mu) = -g k_B T e^{\frac{\mu}{k_BT}} \frac{V}{\lambda_{dB}^3} \pm \frac{1}{2^{5/2}} g k_B T e^{\frac{2\mu}{k_BT}} \frac{V}{\lambda_{dB}^3} + \dots}$$

$$\bar{N} = -\left(\frac{\partial \Phi}{\partial \mu}\right)_{T,V} \rightarrow \text{eliminate } \mu = \mu(T,V,n)$$

$$\boxed{\Phi_{f,b}(T,V,\mu) = -k_B T V n \mp k_B T V \frac{n^2 \lambda_{dB}^3}{2^{7/2} g} + \dots}$$

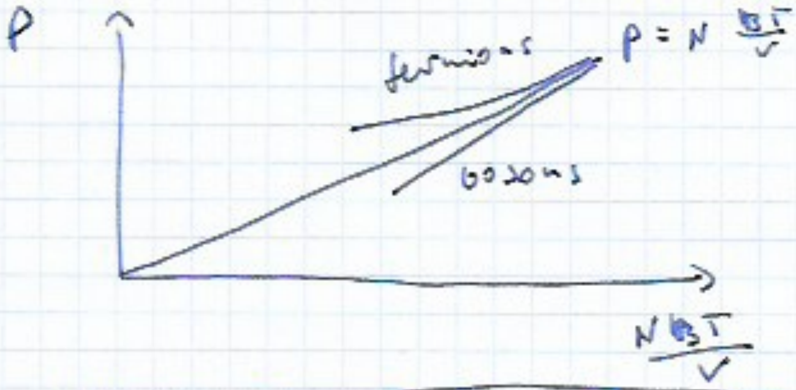
$$\boxed{pV = -V\left(\frac{\partial \Phi_{f,b}}{\partial V}\right)_{T,\mu} = k_B T \underbrace{nV}_{\bar{N}} \pm k_B T V \frac{n^2 \lambda_{dB}^3}{16 g} + \dots}$$

<u>Equation of state</u>

$$\boxed{pV = \underbrace{N k_B T}_{\text{classical}} \underbrace{\overset{f}{\underset{b}{\pm}} N k_B T \left(\frac{n \lambda_{dB}^3}{2^{5/2} g}\right)}_{\text{quantum correction}}}$$

quantum virial expansion

when $T \to \infty$, $\lambda_{dB} \to 0$ and classical gas limit



$p_{fermions} > p_{classical}$

$p_{bosons} < p_{classical}$

At low T fermions and bosons must be treated separately.

§4. GAS OF IDEAL FERMIONS AT T=0

$$\hat{H} = \sum_\sigma \int d^3r \; \hat{\psi}^\dagger_\sigma(\vec r) \left[-\frac{\hbar^2}{2m} \nabla^2 \right] \hat{\psi}_\sigma(\vec r) =$$

$$= \sum_{\vec k \sigma} \underbrace{\frac{\hbar^2 k^2}{2m}}_{\varepsilon_{\vec k}} \hat{a}^\dagger_{\vec k \sigma} \hat{a}_{\vec k \sigma} = \sum_{\vec k \sigma} \varepsilon_{\vec k} \hat{n}_{\vec k \sigma}$$

$\{|\vec k \sigma\rangle\}$ - basis $\qquad \langle \vec r \sigma' | \vec k \sigma \rangle = \frac{1}{\sqrt{V}} e^{i \vec k \cdot \vec r} \delta_{\sigma\sigma'}$

P.B.C. $\qquad \vec k = \frac{2\pi}{L} \vec n$

$\vec n = (n_x, n_y, n_z) \quad n_i \in \mathbb{Z}$

$$\psi(x+L, y, z) = \psi(x, y, z)$$

$$e^{i k_x (x+L)} = e^{i k_x x} \to e^{i k_x L} = 1 = e^{2\pi n i}$$

□

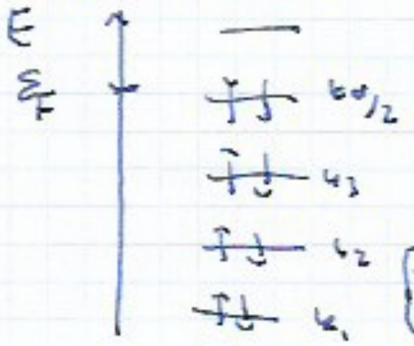
Some order of one particle basis

$|k_1 \uparrow\rangle, |k_1 \downarrow\rangle, |k_2 \uparrow\rangle, |k_2 \downarrow\rangle, \ldots$

$\varepsilon_{k_1} \le \varepsilon_{k_2} \le \varepsilon_{k_3} \le \ldots$

Ground state $\quad T=0$

$$\boxed{|FS\rangle = a^\dagger_{k_1\uparrow} a^\dagger_{k_1\downarrow} a^\dagger_{k_2\uparrow} a^\dagger_{k_2\downarrow} \cdots a^\dagger_{k_{N/2}\uparrow} a^\dagger_{k_{N/2}\downarrow} |vac\rangle}$$



$\boxed{p_F = \hbar k_F}$ Fermi momentum

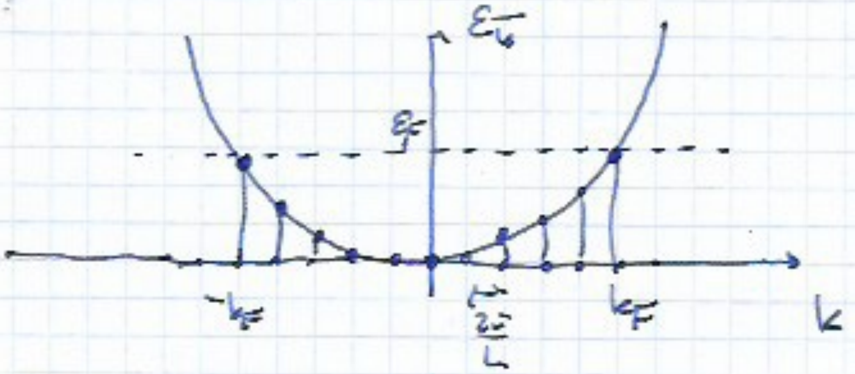
$\boxed{\varepsilon_F = \frac{\hbar^2 k_F^2}{2m}}$ Fermi energy

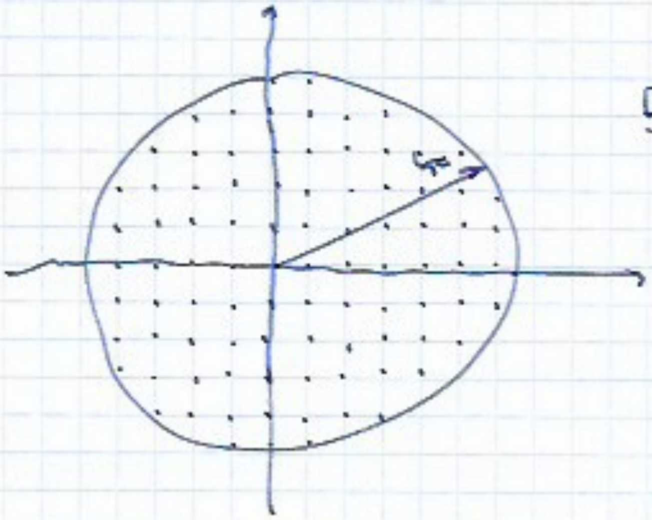
$\boxed{k_F = \frac{1}{\hbar}\sqrt{2m\varepsilon_F}}$ Fermi wave vector

$\boxed{\lambda_F = \frac{2\pi}{k_F}}$ Fermi wave length

$\boxed{T_F = \frac{\varepsilon_F}{k_B}}$ Fermi temperature

$\boxed{v_F = \frac{\hbar k_F}{m}}$ Fermi velocity





Fermi sea

$$\hat{n}_{k\sigma}|FS\rangle = \begin{cases} 1\,|FS\rangle & |k| \le k_F \\ 0\,|FS\rangle & |k| > k_F \end{cases}$$

$\Theta(k_F - |\vec{k}|)$ – step function

Thermodynamic limit $\quad N \to \infty,\ V \to \infty,\ \frac{N}{V} = n = \text{const.}$

$$N = \langle FS|\hat{N}|FS\rangle = \langle FS|\sum_{k\sigma} \hat{n}_{k\sigma}|FS\rangle =$$

$$= \sum_{k\sigma} \langle FS|\hat{n}_{k\sigma}|FS\rangle = \sum_{k\sigma} \Theta(k_F - |\vec{k}|) \overbrace{\langle FS|FS\rangle}^{=1}$$

$$\underset{\text{spin}}{=} 2 \frac{L^3}{(2\pi)^3} \int d^3k\ \Theta(k_F - |\vec{k}|) \underset{\text{[spherical coordinates}}{=}$$

$$= 2 \frac{L^3}{(2\pi)^3} \underbrace{\int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \sin\theta}_{4\pi} \underbrace{\int_0^{k_F} dk\, k^2}_{\frac{1}{3}k_F^3} = \frac{V}{3\pi^2} k_F^3$$

$$\Rightarrow \quad \boxed{n = \frac{N}{V} = \frac{k_F^3}{3\pi^2}} \longleftrightarrow \boxed{k_F = (3\pi^2 n)^{1/3}}$$

e.g. $\boxed{Cu}$ $\quad n \simeq 8.47 \cdot 10^{28}\ 1/m^3$

$k_F = 13.6 \cdot 10^{9}\ 1/m = 13.6\ nm^{-1}$

$\lambda_F \simeq 0.46\ nm$

$\varepsilon_F = \frac{\hbar^2 k_F^2}{2m} = 10^{-18}\ J \simeq 7\,eV$

$T_F = 10^5\ K$

$v_F = 10^6\ m/s = 0.5\%\cdot c$

Metals

	$n = \frac{N}{V}\left[\frac{1}{cm^3}\right]$	$v_F\left[\frac{cm}{s}\right]$	$\varepsilon_F[eV]$	$T_F[K]$
Li	$4.6\cdot10^{22}$	$1.3\cdot10^{8}$	4.7	$5.5\cdot10^{4}$
Na	2.5	1.1	3.1	3.7
K	1.34	0.85	2.1	2.4
Rb	1.08	0.79	1.8	2.1
Cs	0.86	0.73	1.5	1.8
Cu	8.5	1.56	7.0	8.2
Ag	5.76	1.38	5.5	6.4
Au	5.9	1.39	5.5	6.4

$$c = 3\cdot10^{8}\ m/s = 3\cdot10^{10}\ cm/s$$

other systems

matter	particles	$T_F\ [K]$
liquid ^3He	atoms	0.3
metal	electrons	$5\cdot10^{4}$
white dwarfs	electrons	$3\cdot10^{9}$
nuclear matter	nucleus	$3\cdot10^{11}$
neutron stars	neutrons	$3\cdot10^{12}$

As long as $T \ll T_F$ we are in a low-energy regime

$$k_B = 1.38\cdot10^{-23}\ J/K = 8.6\cdot10^{-5}\ \frac{eV}{K}$$

$$1eV = 1.6\cdot10^{-19}\ J$$

$$1eV \Leftrightarrow 10^{4}\ K$$

Simple estimates

$$n^{-1/3} \sim a_B = \frac{\hbar^2}{me^2} \quad \text{Bohr's radius}$$

$$k_F \sim \frac{1}{a_B} \qquad \frac{e^2}{\hbar c} \text{ - fine constant}$$

$$\boxed{\alpha = \frac{e^2}{\hbar c} \simeq \frac{1}{137}}$$

$\Rightarrow$

$$\varepsilon_F \sim \frac{\hbar^2}{m a_B^2} \sim \left(\frac{e^2}{\hbar c}\right)^2 mc^2 \sim \left(\frac{1}{137}\right)^2 mc^2$$

$$p_F \sim \frac{\hbar}{a_B} \sim \left(\frac{e^2}{\hbar c}\right) mc \sim \frac{1}{137} mc$$

$$v_F \sim \frac{p_F}{m} \sim \left(\frac{e^2}{\hbar c}\right) c \sim \frac{1}{137} c$$

Internal energy

$$\boxed{E^{(0)}} = \langle FS | \hat{H} | FS \rangle = \sum_{k\sigma} \frac{\hbar^2 k^2}{2m} \langle FS | \hat{n}_{k\sigma} | FS \rangle =$$

$$= 2 \frac{V}{(2\pi)^3} \frac{\hbar^2}{2m} \int d_3k \, k^2 =$$

$$= 2 \frac{V}{(2\pi)^3} \frac{\hbar^2}{2m} 4\pi \int_0^{k_F} dk \, k^4 = \frac{V}{5\pi^2} \frac{\hbar^2}{2m} k_F^5 =$$

$$= \frac{V}{5\pi^2} \varepsilon_F k_F^3 = \boxed{\frac{3}{5} N \varepsilon_F}$$

$$\varepsilon_F = \frac{\hbar^2}{2m} k_F^2 = \frac{\hbar^2}{2m} (3\pi^2 n)^{2/3}$$

$$\boxed{\frac{E^{(0)}}{N} = \frac{3}{5} \varepsilon_F = \frac{3}{5} \frac{\hbar^2}{2m} (3\pi^2)^{2/3} n^{2/3}}$$

or

$$\boxed{E^{(0)} = \frac{3}{5} \frac{\hbar^2}{2m} (3\pi^2)^{2/3} N^{5/3} V^{-2/3}}$$

pressure

$$p = -\left(\frac{\partial E^{(0)}}{\partial V}\right)_N = -\frac{3}{5}\frac{\hbar^2}{2m}(3\pi^2)^{2/3} N^{5/3}\left(-\frac{2}{3}\right) V^{\overbrace{-2/3-1}^{-5/3}} =$$

$$= \frac{2}{5}\frac{\hbar^2}{2m}(3\pi^2)^{2/3} n^{5/3}$$

$1\,Pa = \frac{1N}{1m^2}$

$1\,atm = 101\,325\ Pa$

Earth sea level pressure

$$\boxed{p = \frac{\hbar^2}{5m}(3\pi^2)^{2/3} n^{5/3}\Big|_{Cu} = 10^6\ atm}$$

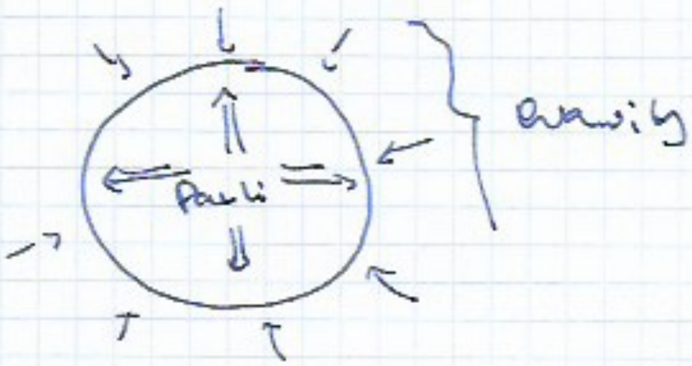
pressure due to Pauli principle

for classical ideal gas

$$p = \frac{N}{V}RT = nRT = \underline{\underline{0}} \quad \text{at } T=0$$

$p>0$ for electrons yields an instability. One has to add Coulomb interaction between e-e and electrons and ions.

White dwarfs the Pauli pressure is balanced by the gravity attraction stopping a colapse.



Ideal fermions in a Zeeman magnetic field

$$H_{zeeman} = -g\,\vec{\mu}\cdot\vec{B}$$ ← uniform external magnetic field

↑ spin magnetic moment of electrons

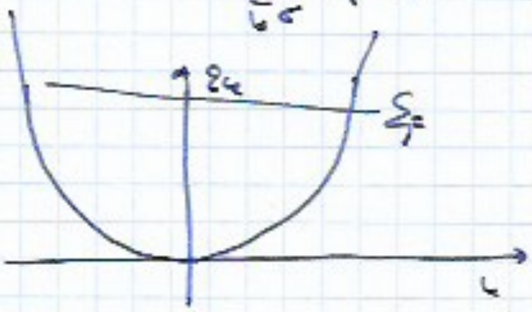
$\vec{B} = (0, 0, B)$

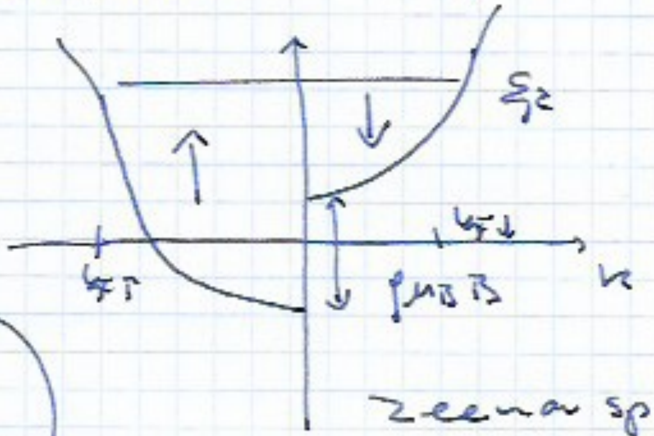
$H_{zeeman} = -g\,\mu_z B \qquad \mu_z = \pm \frac{\mu_B}{2}$ spin magnetic moment for up / down spins

$$\hat{H} = \sum_{k\sigma}\left(\frac{\hbar^2 k^2}{2m} - \sigma g \frac{\mu_B}{2} B\right)\hat{a}^\dagger_{k\sigma}\hat{a}_{k\sigma}$$

$\mu_B = \frac{e\hbar}{2mc}$ Bohr magneton

$$\hat{H} = \sum_{k\sigma}\left(\varepsilon_k - \sigma g \frac{\mu_B}{2} B\right)\hat{n}_{k\sigma} \qquad g = 2$$





Zeeman spin splitting

$$N_\sigma = \frac{V}{(2\pi)^3}\int d^3k\;\theta(k_{F\sigma} - |k|) = V\frac{1}{(2\pi)^3}\cdot 4\pi\int_0^{k_{F\sigma}} dk\,k^2 =$$

$$= V\frac{1}{2\pi^2}\frac{1}{3}k_{F\sigma}^3$$

$$\boxed{n_\sigma = \frac{N_\sigma}{V} = \frac{1}{6\pi^2}k_{F\sigma}^3}$$

density $\quad n = n_\uparrow + n_\downarrow = \frac{1}{6\pi^2}\left(k_{F\uparrow}^3 + k_{F\downarrow}^3\right)$

magnetization $\quad m = \mu_B\frac{n_\uparrow - n_\downarrow}{2} = \frac{\mu_B}{2}\left(\frac{1}{6\pi^2}\right)\left(k_{F\uparrow}^3 - k_{F\downarrow}^3\right)$

Fermi energy

$$\varepsilon_F = \frac{\hbar^2 k_{F\uparrow}^2}{2m} - \frac{g\mu_B B}{2} = \frac{\hbar^2 k_{F\downarrow}^2}{2m} + \frac{g\mu_B B}{2}$$

$$k_{F\uparrow}^2 = \frac{2m}{\hbar^2}\left(\varepsilon_F + \frac{g\mu_B B}{2}\right)$$

$$k_{F\downarrow}^2 = \frac{2m}{\hbar^2}\left(\varepsilon_F - \frac{g\mu_B B}{2}\right)$$

$$m = \frac{\mu_B}{2}\left(\frac{1}{6\pi^2}\right)\left(\frac{2m}{\hbar^2}\right)^{3/2}\left[\left(\varepsilon_F + \frac{g\mu_B B}{2}\right)^{3/2} - \left(\varepsilon_F - \frac{g\mu_B B}{2}\right)^{3/2}\right]$$

at small B we can expand

$$m = \frac{\mu_B}{2}\left(\frac{1}{6\pi^2}\right)\left(\frac{2m}{\hbar^2}\right)^{3/2}\varepsilon_F^{3/2}\left[\left(1 + \frac{g\mu_B B}{2\varepsilon_F}\right)^{3/2} - \left(1 - \frac{g\mu_B B}{2\varepsilon_F}\right)^{3/2}\right] \simeq$$

$$\simeq \frac{\mu_B}{2}\left(\frac{1}{6\pi^2}\right)\left(\frac{2m}{\hbar^2}\right)^{3/2}\varepsilon_F^{3/2}\left[\cancel{1} + \frac{3}{2}\frac{g\mu_B B}{2\varepsilon_F} - \cancel{1} + \frac{3}{2}\frac{g\mu_B B}{2\varepsilon_F} + \dots\right]$$

$$= \frac{\mu_B}{2}\left(\frac{1}{6\pi^2}\right)\left(\frac{2m}{\hbar^2}\right)^{3/2}\varepsilon_F^{3/2}\cdot 3\,\frac{g\mu_B B}{2\varepsilon_F}$$

Pauli susceptibility $\chi = \left.\frac{m}{B}\right|_{B\to 0}$

$$\chi = 3\frac{\mu_B}{2}\left(\frac{1}{6\pi^2}\right)\left(\frac{2m}{\hbar^2}\right)^{3/2}\varepsilon_F^{3/2}\,\frac{g\mu_B}{2\varepsilon_F}$$

$$\boxed{\chi = 3\,\frac{g}{4}\,\mu_B^2\left(\frac{2m}{\hbar^2}\right)^{3/2}\varepsilon_F^{1/2}} \qquad \mu_B = \frac{e\hbar}{2mc}$$

$$\varepsilon_F = \frac{\hbar^2}{2m}(3\pi^2 n)^{2/3}$$

$$\chi \sim \mu_B^2\left(\frac{m}{\hbar^2}\right)^{3/2}\cdot\varepsilon_F^{1/2} = \frac{e^2\hbar^2}{2m^2c^2}\left(\frac{m}{\hbar^2}\right)^{3/2}\sqrt{\frac{\hbar^2}{2m}}\,(3\pi^2 n)^{1/3} \simeq$$

$$\simeq \frac{e^2\hbar^2}{m^2c^2}\,\frac{m^{3/2}}{\hbar^3}\,\frac{\hbar}{m}\,n^{1/3} \simeq O(\hbar^0)$$

why no $\hbar$ present?

Van Leeuwen theorem – in classical physics $M \xrightarrow[\hbar \to 0]{} 0$ and $\chi \xrightarrow[\hbar \to 0]{} 0$

<u>What is wrong?</u>

Limits ·) $k_B T \ll \varepsilon_F = \frac{\hbar^2 n^{2/3}}{m} \to 0$

$\quad k_B T \downarrow 0$ ok

··) $\mu_B B \ll \varepsilon_F$

$$\frac{e\hbar}{mc} B \ll \frac{\hbar^2}{m} n^{2/3}$$

or $$B \ll \frac{\hbar}{e} c n^{2/3} \xrightarrow[\hbar \to 0]{} 0$$

In a constant magnetic field one <u>can not</u> take <u>$\hbar \to 0$ limit !</u>

Ideal fermions - some results in d-dimensions

$$N = 2\,\frac{V}{(2\pi)^d}\int_0^{k_F} d_d k = 2\,\frac{V}{(2\pi)^d}\int d\Omega_d \int_0^{k_F} dk\, k^{d-1}$$

$$\int d\Omega_d = \frac{d\,\pi^{d/2}}{\Gamma\left(\frac{d}{2}+1\right)}$$

$$n = \frac{N}{V} = \frac{2}{(2\pi)^d}\cdot\frac{\cancel{d}\,\pi^{d/2}}{\Gamma\left(\frac{d}{2}+1\right)}\,\frac{1}{\cancel{d}}\,k_F^{\,d} = 2^{1-d}\,\pi^{\overbrace{d/2-d}^{-d/2}}\,\frac{k_F^{\,d}}{\underbrace{\Gamma\left(\frac{d}{2}+1\right)}_{\frac{d}{2}\Gamma\left(\frac{d}{2}\right)}}$$

$$\boxed{k_F = \left(2^{d-1}\pi^{d/2}\cdot\frac{d}{2}\Gamma\left(\frac{d}{2}\right)\,n\right)^{1/d}}$$

$d=3$ $\quad \Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2}$, $\quad k_F = \left(\cancel{2}^{2}\,\pi^{3/2}\cdot\frac{3}{\cancel{2}}\cdot\frac{\pi^{1/2}}{\cancel{2}}\cdot n\right)^{1/3} =$

$\Gamma(1/2)=\sqrt{\pi}$, $\Gamma(\alpha+1)=\alpha\Gamma(\alpha)=\alpha!$

$$= \left(3\pi^2\cdot n\right)^{1/3}$$

$$\boxed{\varepsilon_F = \frac{\hbar^2}{2m}\left(2^{d-1}\pi^{d/2}\cdot\frac{d}{2}\cdot\Gamma\left(\frac{d}{2}\right)n\right)^{2/d}}$$

$$E^{(0)} = 2\,\frac{V}{(2\pi)^d}\,\frac{\hbar^2}{2m}\cdot\frac{d\,\pi^{d/2}}{\Gamma\left(\frac{d}{2}\right)\cdot\frac{d}{2}}\cdot\int_0^{k_F} dk\, k^{d+1} =$$

$$= 2\,\frac{V}{(2\pi)^d}\,\frac{\hbar^2}{\cancel{2}m}\,\frac{\cancel{d}\,\pi^{d/2}}{\Gamma\left(\frac{d}{2}\right)\cdot\cancel{\frac{d}{2}}}\cdot\frac{1}{d+2}\,k_F^{\,d+2}$$

$$\frac{E^{(0)}}{V} = 2^{1-d}\,\pi^{-d}\,\frac{\hbar^2}{\cancel{2}m}\,\pi^{d/2}\,\frac{1}{\Gamma\left(\frac{d}{2}\right)}\,\frac{k_F^{\,d+2}}{d+2} =$$

$$= 2^{2-d}\,\pi^{-d/2}\,\frac{\hbar^2}{2m}\,\frac{1}{\Gamma\left(\frac{d}{2}\right)}\,\frac{k_F^{\,d+2}}{d+2}$$

$$N = V\, 2^{1-d}\, \pi^{-d/2}\, \frac{k_F^{\,d}}{\frac{d}{2}\Gamma\left(\frac{d}{2}\right)}$$

$$k_F^{\,d} = 2^{d-1}\, \pi^{d}\, \frac{d}{2}\Gamma\left(\frac{d}{2}\right) \frac{N}{V}$$

$$\varepsilon_F = \frac{\hbar^2}{2m}\left(2^{d-1}\, \pi^{d/2}\, \frac{d}{2}\Gamma\left(\frac{d}{2}\right)\cdot \frac{N}{V}\right)^{2/d}$$

$$E^{(0)} = V\, 2^{2-d}\, \pi^{-d/2}\, \frac{\hbar^2}{2m}\, \frac{1}{\Gamma\left(\frac{d}{2}\right)}\, \frac{k_F^{\,d+2}}{d+2}$$

$$\frac{E^{(0)}}{N} = \frac{V\, 2^{2-d}\, \pi^{-d/2}\, \frac{\hbar^2}{2m}\, \frac{1}{\Gamma(d/2)}\, \frac{k_F^{\,d+2}}{d+2}}{V\, 2^{1-d}\, \pi^{-d/2}\, \frac{k_F^{\,d}}{\frac{d}{2}\Gamma\left(\frac{d}{2}\right)}} =$$

$$= 2\, \underbrace{\frac{\hbar^2}{2m} k_F^{\,2}}_{\varepsilon_F} \cdot \frac{d/2}{d+2} = \frac{d}{d+2}\, \varepsilon_F$$

arxiv: 1008.1306

$$\boxed{\frac{E^{(0)}}{N} = \frac{d}{d+2}\, \varepsilon_F \xrightarrow[d\to\infty]{} \varepsilon_F}$$

In $d=\infty$ limit all fermions have the Fermi energy ε_F and $k_F = \frac{\sqrt{2m\varepsilon_F}}{\hbar}$!