WHAT IS DIMENSION OF BRA AND KET?

based on 2005.03187

SI unit system

time	[s]	second	$\to T$
length	[m]	meter	$\to L$
mass	[kg]	kilogram	$\to M$
current	[A]	amper	
temperature	[K]	kelvin	
luminosity	[cd]	candela	

Any physical quantity is expected to be characterized by a unit or dimension

So what is $[|\psi\rangle]$ or $[|\vec{r}\rangle]$?

A physical quantity Q has a dimension $[Q]$
If $[Q]=1$ then we say that Q is dimensionless

Examples

$[\vec{r}] = L$ – position

$[\vec{p}] = M L T^{-1}$ – momentum

$[\hbar] = [\vec{r}]\cdot[\vec{p}] = M L^2 T^{-1}$ Planck constant

$\square$

How about $|\alpha\rangle$ or $|\beta\rangle$ and $\langle\alpha|$ or $\langle\beta|$?

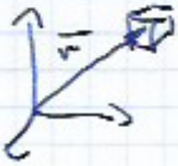
<u>wave function</u> $\mathcal{H} \ni \Psi(\vec{r},t)$ – probability amplitude

$\llcorner$ <u>Hilbertspace</u>

probability density $\rho(\vec{r},t) = |\Psi(\vec{r},t)|^2 = \Psi^*(\vec{r},t)\,\Psi(\vec{r},t)$

Probability to find a particle at $\vec{r}$

d-dimensional space



$$P(\vec{r},t) = \rho(\vec{r},t)\, d_d r =$$
$$= |\Psi(\vec{r},t)|^2\, d_d r$$

Note $[P(\vec{r},t)] = [1]$ dimension-less!

$$\rightarrow [\rho(\vec{r},t)] = \left[\frac{1}{L^d}\right]$$

$$\rightarrow [\Psi(\vec{r},t)] = \left[\frac{1}{L^{d/2}}\right]$$

<u>Dirac bra-kets</u>

$a_n \in \mathbb{R}$

$\hat{A}\,|a_n\rangle = a_n\,|a_n\rangle$ discrete eigenstates (basis)

$\hat{A}\,|a\rangle = a|a\rangle$ continuous eigenstates (basis)

$\hat{A}$ - observable, operator $a \in \mathbb{R}$

must be Hermitian (self-adjoint)

$|a\rangle$ and $|a_n\rangle$ are elements of a vector space V with a scalar product such that

$$\langle a|b\rangle \in \mathbb{C} \quad \text{or} \quad \langle a_n|b_m\rangle \in \mathbb{C}$$

$|a_n\rangle$ - column vector $\in V$ $\begin{pmatrix} a_1 \\ a_2 \\ \vdots \end{pmatrix} \rightarrow V \simeq \mathcal{H}$

$\langle a_n|$ - ~~t~~ row vector $(a_1^* \; a_2^* \cdots) \in V^*$ dual space

$\hat{A} : V \longrightarrow V$ $\qquad V^* \simeq \mathcal{H}^* \simeq \mathcal{H}$

$$\hat{A}^+ : V^* \to V^*$$ <u>Hermitian conjugation</u>

If we know $\hat{A}$ then $\hat{A}^+$ is found by demanding

$$\langle \hat{A}^+ \alpha | \beta \rangle = \langle \alpha | \hat{A} \beta \rangle$$

for any $|\alpha\rangle$ and $|\beta\rangle$.

Hermitian operator $\hat{A} = \hat{A}^+$
(self-adjoint operator $\hat{A} = \hat{A}^+$ and the same domains)

<u>Example</u>

$$\hat{\vec{r}} \, |\vec{r}\rangle = \vec{r} \, |\vec{r}\rangle$$ position operator

$$\hat{\vec{p}} \, |\vec{p}\rangle = \vec{p} \, |\vec{p}\rangle$$ momentum operator

Note that $|\vec{r}\rangle$ or $|\vec{p}\rangle$ are not elements of $\mathcal{H}$!

The wave function

$$\Psi(\vec{r}, t) = \langle \vec{r} | \Psi(t) \rangle = \langle \Psi(t) | \vec{r} \rangle^*$$

scalar product

$|\Psi(t)\rangle$ - time dependent <u>state vector</u> of a system.

<u>Dimension of a scalar product?</u>

$$[\langle \alpha | \beta \rangle] = [\langle \alpha |][| \beta \rangle]$$

$$\langle \alpha | \beta \rangle = \langle \beta | \alpha \rangle^* \to [\langle \alpha | \beta \rangle] = [\langle \beta | \alpha \rangle]$$

Complex conjugation does not change the dimension!

In general $[|\alpha\rangle] \neq [\langle\alpha|]$!

Let $|\Psi(t)\rangle$ is a physical state.

We denote $[|\Psi(t)\rangle] = K$ (ket)

$[\langle\Psi(t)|] = B$ (bra)

A physical state is always normalizable !

$$\langle\Psi(t)|\Psi(t)\rangle = 1 \rightarrow [\langle\Psi(t)|] = [|\Psi(t)\rangle]^{-1} \quad \text{or}$$

unitary evolution in time !

$$\boxed{B = K^{-1}}$$

We note ~~[...]~~

$$[\langle\bar{r}|] = r_B$$

$$[|\bar{r}\rangle] = r_K$$

From equation $\langle\bar{r}|\Psi(t)\rangle = \Psi(\bar{r},t)$ and normalization $[\Psi(\bar{r},t)] = L^{-d/2}$ we get

$$r_B \cdot K = r_K \cdot K^{-1} = L^{-d/2} \quad (*)$$

orthonormality of the $\{|\bar{r}\rangle\}$ base

$$\langle\bar{r}|\bar{r}'\rangle = \delta_{(d)}(\bar{r}-\bar{r}') \quad - \text{Dirac } \delta\text{-function}$$

completness of the $\{|\bar{r}\rangle\}$ base

$$\int d_d r\, |\bar{r}\rangle\langle\bar{r}| = \hat{\mathbb{1}}$$

operator

unity operator in infinite dimensional continuous space

$\mathbb{1}^+ = \mathbb{1}$

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Decompose of $|\psi\rangle$ vector

$$|\psi(t)\rangle = \hat{1}\,|\psi(t)\rangle = \int d_d r \;\; |\bar{r}\rangle\langle\bar{r}|\psi(t)\rangle =$$
$$= \int d_d r \;\; \psi(\bar{r},t)\,|\bar{r}\rangle$$

Hence we get

$$K = L^d \cdot L^{-d/2}\, r_K = L^{d/2}\, r_K \rightarrow \boxed{r_K = K L^{-d/2}}$$

which is the same as (*)

Similarly

$$\langle\psi(t)| = \int d_d r \langle\psi(t)|\bar{r}\rangle\langle\bar{r}| = \int d_d r \;\psi(\bar{r},t)^* \langle\bar{r}|$$
$$B = L^d\; L^{-d/2}\, r_B = L^{d/2}\, r_B \rightarrow \boxed{r_B = B\, r^{-d/2}}$$

The same as (*)

$$\rightarrow r_B\, r_K = \underbrace{B\,K}_{B = K^{-1}}\, r^{-d/2}\, r^{-d/2} = r^{-d}$$

Consider momentum space

$$\langle\bar{p}|\psi(t)\rangle = \langle\psi(t)|\bar{p}\rangle^* = \Phi(\bar{p},t)$$

$[\langle\bar{p}|] = P_B$ + normalization of $|\psi\rangle$ implies
$[\,|\bar{p}\rangle] = P_K$ normalization of $\Phi(\bar{p},t)$

$$|\Phi(t)\rangle = \int d_d p \;\; |\bar{p}\rangle\langle\bar{p}|\Phi(t)\rangle = \int d_d p\; \Phi(\bar{p},t)\,|\bar{p}\rangle$$
$$\langle\psi(t)| = \int d_d p \langle\psi(t)|\bar{p}\rangle\langle\bar{p}| = \int d_d p \;\; \Phi^*(\bar{p},t)\, \langle\bar{p}|$$

where $\langle\bar{p}|\bar{p}'\rangle = \delta_d(\bar{p}-\bar{p}')$ $\qquad [\Phi] = (MLT^{-1})^{-d/2}$

normalization

$$\int d_d p \;\; |\bar{p}\rangle\langle\bar{p}| = \hat{1}$$

$[\,|\bar{p}\rangle] = P_K$, $[\langle\bar{p}|] = P_B$

$$\rightarrow \quad K = (MLT^{-1})^{d} \cdot (MLT^{-1})^{-d/2} \, P_k$$

$$\rightarrow \quad P_B K = P_k \underbrace{K^{-1}}_{B} = (MLT^{-1})^{-d/2}$$

It does not bring any new information!

Another try: A link between $\Psi(\vec r, t)$ and $\Phi(\vec p, t)$

$$\Phi(\vec p, t) = \langle \vec p \,|\, \underset{\hat{1}}{\uparrow} \, \Psi(t) \rangle =$$

$$= \int d^d r \, \langle \vec p | \vec r \rangle \langle \vec r | \Psi(t) \rangle =$$

$$= \int d^d r \, \langle \vec p | \vec r \rangle \, \Psi(\vec r, t)$$

where $\langle \vec p | \vec r \rangle = \langle \vec r, \vec p \rangle^{*} = \dfrac{e^{-i \frac{\vec p \cdot \vec x}{\hbar}}}{(2\pi\hbar)^{d/2}}$

a postulate or derived from $[\hat x_i, \hat p_j] = i\hbar\delta_{ij}$

no new result!

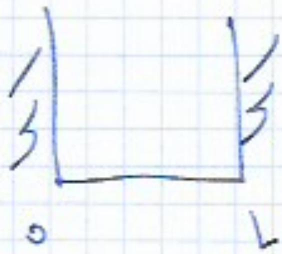
Expand $|\Psi(t)\rangle$ in a discrete base $\{|n\rangle\}$

$\langle n|n'\rangle = \delta_{nn'}$ orthonormality

$\sum_n |n\rangle\langle n| = \mathbb{1}$ completeness

<u>Example</u>

infinite quantum well



$$\Psi_n(x,t) = \sqrt{\frac{2}{L}}\, e^{-i E_n t/\hbar} \sin\left(x \frac{n\pi}{L}\right)$$

$$E_n = n^2 \frac{\hbar^2}{2mL^2} \qquad n = 1, 2, 3, \ldots$$

$\square$

$$|\Psi(t)\rangle = \hat{\mathbb{1}}\, |\Psi(t)\rangle = \sum_n |n\rangle\langle n|\Psi(t)\rangle =$$

$$= \sum_n c_n(t)\, |n\rangle$$

another wave function

$p_n = |c_n(t)|^2$ – probability to find a particle in a state n

normalization

$$\langle \Psi|\Psi\rangle = \sum_{nm}^{\infty} c_m(t)^* c_n(t) \langle m|n\rangle = 1$$

hence $\quad [c_m^*(t)]\,[c_n(t)]\,[\langle n|m\rangle] = 1$

because $[|n\rangle] = N_K$, $[\langle n|] = N_B$ $\quad$ $[\langle n|m\rangle] = 1$ orthonormality $\quad N_B N_K = 1$

Thus $[c_n(t)] = 1$

(complex conjugation does not change a dimension)

And we see

$$K = N_K$$

and similarly

$$B = N_B = N_K^{-1} = K^{-1}$$

$\Rightarrow$ $\boxed{\text{There are } \underline{\text{no}} \text{ condition to fix a dimension of } |\Psi\rangle}$

We have to fix some convention:

1) $\boxed{[\,|\Psi\rangle\,] = K = 1}$

2) $[\langle\alpha|] = [\,|\alpha\rangle]^* \Rightarrow \boxed{[\langle\Psi|] = 1}$

Hence we get

$$K = B = N_K = N_B = 1, \quad r_B = r_K = L^{-d/2},$$
$$p_B = p_K = (MLT^{-1})^{-d/2}$$

<u>Summary:</u>

$$\boxed{\begin{aligned}
&[\langle\alpha|\beta\rangle] = [\langle\alpha|][\,|\beta\rangle] \\
&[\langle\alpha|] = [\,|\alpha\rangle] \\
&[\,|\Psi\rangle] = 1 \quad \text{physical state} \\
&[\,|\vec{r}\rangle] = L^{-d/2} \\
&[\,|\vec{p}\rangle] = (MLT^{-1})^{-d/2} \\
&[\,|n\rangle] = 1
\end{aligned}}$$

(P)