2-point correlation function

$$\hat{H} = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2$$

$$G(\tau) = \frac{1}{Z} \operatorname{Tr}\left[e^{-\beta \hat{H}} \hat{x}(\tau)\, \hat{x}(0) \right]$$

$$\hat{x}(\tau) = e^{\frac{\hat{H}\tau}{\hbar}}\, \hat{x}\, e^{-\frac{\hat{H}\tau}{\hbar}} \quad , \; 0 < \tau < \beta\hbar$$

Show, using standard QM that

$$G(\tau) = \frac{\hbar}{2m\omega} \frac{\cosh\left[\left(\frac{\beta\hbar}{2} - \tau\right)\omega\right]}{\sinh\left[\frac{\beta\hbar\omega}{2}\right]}$$

Path integral

$$G(\tau) = \frac{\int_{x(\beta\hbar) = x(0)} D[x(\tau)]\, x(\tau)\, x(0)\, e^{-S_E/\hbar}}{\int_{x(\beta\hbar) = x(0)} D[x(\tau)]\, e^{-S_E/\hbar}}$$

The c' factor drops out!

$$x(\tau) = \frac{1}{\beta}\left[a_0 + \sum_{k=1}^{\infty} \left[(a_k + i b_k) e^{i\omega_k \tau} + (a_k - i b_k) e^{-i\omega_k \tau} \right] \right]$$

$$x(0) = \frac{1}{\beta}\left[a_0 + \sum_{\ell=1}^{\infty} 2 a_\ell \right]$$

$$\boxed{G(\tau) = \langle x(\tau)\, x(0) \rangle \overset{(*)}{=} \frac{\int da_0 \int \prod_{n\geq 1} da_n\, db_n\, x(\tau)\, x(0)\, e^{-S_E/\hbar}}{\int da_0 \int \prod_{n \geq 1} da_n\, db_n\, e^{-S_E/\hbar}}}$$

since, $a_0, a_n, b_n \in \mathbb{R}$ and S_E is Gaussian

$$\langle a_0 a_k \rangle = \langle a_0 b_k \rangle = \langle a_k b_\ell \rangle = 0$$

$$\langle a_k a_\ell \rangle \approx \langle b_k b_\ell \rangle \approx \delta_{k\ell}$$

$\langle \dots \rangle$ in sense of $(*)$

• $G(\tau) = \frac{1}{\beta^2}\left\langle a_0^2 + \sum_{k=1}^{\infty} 2a_k^2\left(e^{i\omega_k\tau} + e^{-i\omega_k\tau}\right)\right\rangle$

where

$$\boxed{\langle a_0^2\rangle} = \frac{\int da_0\, a_0^2\, e^{-\frac{m}{2\beta}\omega^2 a_0^2}}{\int da_0\, e^{-\frac{m}{2\beta}\omega^2 a_0^2}} =$$

$$= -\frac{2}{m\omega^2}\frac{\partial}{\partial(1/\beta)}\ln\int da_0\, e^{-\frac{m}{2\beta}\omega^2 a_0^2} =$$

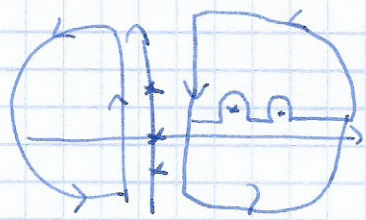
$$= -\frac{2}{m\omega^2}\frac{\partial}{\partial(1/\beta)}\ln\sqrt{\frac{2\pi\beta}{m\omega^2}} = \boxed{\frac{\beta}{m\omega^2}}$$

$$\boxed{\langle a_k^2\rangle} = \frac{\int da_k\, a_k^2\, e^{-\frac{m}{\beta}(\omega_k^2+\omega^2)a_k^2}}{\int da_k\, e^{-\frac{m}{\beta}(\omega_k^2+\omega^2)a_k^2}} = \boxed{\frac{\beta}{2m(\omega_k^2+\omega^2)}}$$

Thus we get

$$G(\tau) = \frac{1}{m\beta}\left(\frac{1}{\omega^2} + \sum_{k=1}^{\infty}\frac{e^{i\omega_k\tau} + e^{-i\omega_k\tau}}{\omega_k^2+\omega^2}\right) = \frac{1}{m\beta}\sum_{k=-\infty}^{\infty}\frac{e^{i\omega_k\tau}}{\omega_k^2+\omega^2}$$

Using a standard sum over Matsubara frequency (lecture)



$$\sum_n \cancel{H}h(i\omega_n) = -\frac{\beta}{2\pi i}\oint_C \frac{h(z)}{e^{\beta z}-1}\,dz$$

$$\oint f(z)\,dz = 2\pi i\sum_i \mathrm{Res}\, f(z_i)$$

we get

• $$\boxed{G(\tau) = \frac{\hbar}{2m\omega}\,\frac{\cosh\left[\left(\frac{\beta\hbar}{2}-\tau\right)\omega\right]}{\sinh\left[\frac{\beta\hbar\omega}{2}\right]}}$$