Factorization method for discrete set of energies

Find ladder operators $\hat{\eta}_i$ and real constants E_i such that

$$(*)\quad \begin{cases} \hat{\eta}_1^+ \hat{\eta}_1 + E_1 = \hat{H} \longrightarrow \boxed{\text{factorization of Hamiltonian}} \\ \hat{\eta}_2^+ \hat{\eta}_2 + E_2 = \hat{\eta}_1 \hat{\eta}_1^+ + E_1 \\ \hat{\eta}_3^+ \hat{\eta}_3 + E_3 = \hat{\eta}_2 \hat{\eta}_2^+ + E_2 \\ \vdots \end{cases}$$

$$(**)\quad \hat{\eta}_{j+1}^+ \hat{\eta}_{j+1} + E_{j+1} = \hat{\eta}_j \hat{\eta}_j^+ + E_j \qquad j = 1, 2, 3, \ldots$$

Theorem: suppose that $(*)$ is satisfied and that each $\hat{\eta}_i$ has an eigenvector $|\xi_i\rangle$ with eigenvalue zero

$$\hat{\eta}_i |\xi_i\rangle = 0$$

Then: i) The constant E_j is the jth eigenvalue of $\hat{H}$ (arrange in ascending order)

ii) The corresponding eigenvector (up to normalization) is

$$|E_j\rangle = \hat{\eta}_1^+ \hat{\eta}_2^+ \cdots \hat{\eta}_{j-1}^+ |\xi_j\rangle$$

Proof Let $\boxed{\hat{\Lambda}_j \equiv \hat{\eta}_j^+ \hat{\eta}_j + E_j}$

Therefore $\boxed{\hat{\Lambda}_{j+1}} = \hat{\eta}_{j+1}^+ \hat{\eta}_{j+1} + E_{j+1} = \boxed{\hat{\eta}_j \hat{\eta}_j^+ + E_j}$ from $(**)$

We check $\boxed{\hat{\Lambda}_{j+1} \hat{\eta}_j = \hat{\eta}_j \hat{\eta}_j^+ \hat{\eta}_j + E_j \hat{\eta}_j = \hat{\eta}_j (\hat{\eta}_j^+ \hat{\eta}_j + E_j) = \hat{\eta}_j \hat{\Lambda}_j}$

$\boxed{\hat{\Lambda}_j \hat{\eta}_j^+} = \hat{\eta}_j^+ \hat{\eta}_j \hat{\eta}_j^+ + E_j \hat{\eta}_j^+ = \hat{\eta}_j^+ (\hat{\eta}_j \hat{\eta}_j^+ + E_j) = \boxed{\hat{\eta}_j^+ \hat{\Lambda}_{j+1}}$

$\hat{H} = \hat{\Lambda}_1$

$$\hat{H}|E_j\rangle = \hat{\Lambda}_1 \hat{\eta}_1^+ \hat{\eta}_2^+ \cdots \hat{\eta}_{j-1}^+ |\xi_j\rangle =$$

$$= \hat{\eta}_1^+ \hat{\Lambda}_2 \hat{\eta}_2^+ \cdots \hat{\eta}_{j-1}^+ |\xi_j\rangle =$$

$$\vdots$$

$$= \hat{\eta}_1^+ \hat{\eta}_2^+ \cdots \hat{\eta}_{j-1}^+ \hat{\Lambda}_j |\xi_j\rangle$$

However, $\hat{\Lambda}_j |\xi_j\rangle = (\hat{\eta}_j^+ \underbrace{\hat{\eta}_j + E_j) |\xi_j\rangle}_{=0} = E_j |\xi_j\rangle$

Therefore,

$$\hat{H}|E_j\rangle = E_j \hat{\eta}_1^+ \hat{\eta}_2^+ \cdots \hat{\eta}_{j-1}^+ |\xi_j\rangle = E|E_j\rangle$$

We need to show that $E_1 \leq E_2 \leq E_3 \cdots$

Consider $E_{j+1} - E_j$ and assume that $|\xi_{j+1}\rangle$ is normalized

$E_{j+1} - E_j = \langle \xi_{j+1} | E_{j+1} - E_j | \xi_{j+1}\rangle =$ from $(**)$

$$= \langle \xi_{j+1} | \hat{\eta}_j \hat{\eta}_j^+ - \hat{\eta}_{j+1}^+ \underbrace{\hat{\eta}_{j+1} | \xi_{j+1}\rangle}_{=0} =$$

$$= \langle \xi_{j+1} | \hat{\eta}_j \hat{\eta}_j^+ | \xi_{j+1}\rangle = \| \hat{\eta}_j^+ |\xi_{j+1}\rangle \| \geqslant 0$$

Hence $E_{j+1} - E_j \geqslant 0 \Rightarrow E_1 \leq E_2 \leq E_3 \leq \cdots$

We need to show that if E is eigenvalue of $\hat{H}$ then E cannot lie between E_j and E_{j+1}.

Consider an auxiliary vector

$$|\xi_n\rangle = \hat{\eta}_n \hat{\eta}_{n-1} \cdots \hat{\eta}_2 \hat{\eta}_1 |E\rangle$$

Then

$$0 \le \langle \xi_1 | \xi_1 \rangle = \left(\langle E | \hat{\eta}_1\right)^* \hat{\eta}_1 |E\rangle = \langle E | \hat{\eta}_1^+ \hat{\eta}_1 |E\rangle =$$

$$= \langle E | \underbrace{\hat{A} - E_1 | E}_{E|E\rangle} \rangle = E - E_1 \Rightarrow \underline{E \geq E_1}$$

$$0 \le \langle \xi_2 | \xi_2 \rangle = \langle E | \hat{\eta}_1^+ \hat{\eta}_2^+ \hat{\eta}_2 \hat{\eta}_1 |E\rangle =$$

$$= \langle E | \hat{\eta}_1^+ (\hat{\lambda}_2 - E_2) \hat{\eta}_1 |E\rangle =$$

$$= \langle E | \hat{\eta}_1^+ \hat{\eta}_1 (\underbrace{\hat{\lambda}_1 - E_2) |E}_{E|E_1} \rangle =$$

$$= (E - E_2) \langle E | \hat{\eta}_1^+ \hat{\eta}_1 |E\rangle = (E - E_2)(E - E_1)$$

$$\Rightarrow E \geq E_2 \quad \text{or} \quad E = E_1 ,$$

In general

$$0 \le \langle \xi_n | \xi_n \rangle = (E - E_n)(E - E_{n-1}) \cdots (E - E_2)(E - E_1)$$

$$\Rightarrow \quad E > E_n^{max} \quad \text{or} \quad E = E_j , j = 1, \dots, n$$

$\square$

How to find $\hat{\eta}_1$ such that

$$\hat{H} = \hat{\eta}_1^+ \hat{\eta}_1 + E_1 \quad ?$$

Take an Ansatz $\quad \hat{H} = \frac{\hat{p}^2}{2m} + V(x) = \hat{\eta}_1^+ \hat{\eta}_1 + E_1$

$$\hat{\eta}_j = \frac{1}{\sqrt{2m}} \left(\hat{p} + i f_j(x) \right) \qquad f_j(x) \in \mathbb{R}$$

Since $\quad [f(x), \hat{p}] = i\hbar \frac{df}{dx} = f(x)\hat{p} - \hat{p} f(x)$

we get

$$\hat{\eta}_j^+ \hat{\eta}_j = \frac{1}{2m}\left(\hat{p} - i f_j(x)\right)\left(\hat{p} + i f_j(x)\right) =$$

$$= \frac{1}{2m}\left(\hat{p}^2 + i\hat{p} f_j(x) - i f_j(x)\hat{p} + f_j(x)^2\right) =$$

$$= \frac{1}{2m}\left(\hat{p}^2 + \cancel{i f_j(x)\hat{p}} - i^2\hbar \frac{df}{dx} + f_j(x)^2 - i f_j \cancel{(x)\hat{p}}\right) =$$

$$= \frac{1}{2m}\hat{p}^2 + \frac{\cancel{\hbar}\,1}{2m} f_j(x)^2 + \frac{\hbar}{2m}\frac{df_j}{dx}$$

$$\hat{\eta}_j \hat{\eta}_j^+ = \frac{1}{2m}\left(\hat{p} + i f_j\right)\left(\hat{p} - i f_j\right) =$$

$$= \frac{1}{2m}\left(\hat{p}^2 - i\hat{p} f_j + i \underbrace{f_j \hat{p}}_{\hat{p} f + i\hbar f'} + f_j^2\right) =$$

$$= \frac{1}{2m}\left(\hat{p}^2 - i\cancel{\hat{p} f_j} + i\cancel{\hat{p} f_j} - \hbar\frac{df}{dx} + f_j^2\right) =$$

$$= \frac{1}{2m}\hat{p}^2 - \frac{\hbar}{2m} + \frac{1}{2m} f_j(x)^2 .$$

Hence $\quad \hat{\eta}_1^+ \hat{\eta}_1 + E_1 = \frac{\hat{p}^2}{2m} + V(x)$

$$\cancel{\frac{1}{2m}\hat{p}^2} + \frac{1}{2m} f_1(x)^2 + \frac{\hbar}{2m}\frac{df_1}{dx} = \cancel{\frac{\hat{p}^2}{2m}} + V(x)$$

$$\frac{\hbar}{2m}\frac{df_1}{dx} + \frac{1}{2m} f_1(x)^2 = V(x)$$

We need to solve it for $f_1(x)$ and then find others $f_j(x)$ recursively.