Factorization method for harmonic oscillator

$$\frac{1}{2m} f_1(x)^2 + \frac{\hbar}{2m}\frac{df_1(x)}{dx} + E_1 = \frac{1}{2} m\omega^2 x^2$$

particular solution is $f_1(x) = \pm m\omega x$, $E_1 = \mp \frac{1}{2}\hbar\omega$
(no general solution needed since we need to get $\hat{\eta}_1 |\xi_1\rangle = 0$)

Sign $\to$ E_j must be maximal, hence,

$$\boxed{f_1(x) = -m\omega x, \qquad E_1 = \frac{\hbar\omega}{2} \text{ - ground state}}$$

Now
$$\hat{\eta}_2^+ \hat{\eta}_2 + E_2 = \hat{\eta}_1 \hat{\eta}_1^+ + E_1$$

$$\cancel{\frac{1}{2m}\hat{p}^2} + \frac{1}{2m} f_2^2 + \frac{\hbar}{2m}\frac{df_2}{dx} = \cancel{\frac{1}{2m}\hat{p}^2} + \frac{1}{2m}\underbrace{(-m\omega x)}_{f_1}^2 - \frac{\hbar}{2m}\frac{d}{dx}\overbrace{(-m\omega x)}^{f_1}$$

$$\frac{1}{2m} f_2^2 + \frac{\hbar}{2m}\frac{df_2}{dx} = \frac{1}{2} m\omega^2 x^2 + \hbar\omega$$

solution $\boxed{f_2 = -m\omega x, \quad E_2 = \frac{3}{2}\hbar\omega}$

Similarly we always get

$$\boxed{f_j(x) = -m\omega x, \qquad E_j = (j - \tfrac{1}{2})\hbar\omega}$$

So all $\hat{\eta}_j$ operators are the same

$$\boxed{\hat{\eta}_j = \frac{1}{\sqrt{2m}}(\hat{p} - im\omega\hat{x})} \qquad |\xi_1\rangle \text{ - ground state}$$

and $|E_j\rangle \sim \underbrace{(\hat{p} + im\omega\hat{x})^{j-1}}_{(\hat{a}^+)^{j-1}} |\xi_1\rangle$ $\qquad$ E_1 - ground state energy

with $\underbrace{(\hat{p} - im\omega\hat{x})}_{\hat{a}} |\xi_1\rangle = 0$ $\qquad$ the same for all j

$\square$