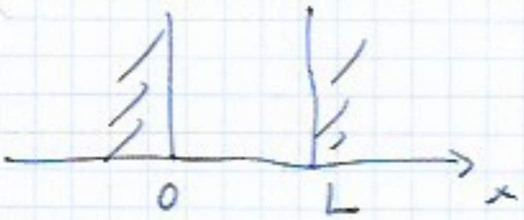
Factorization method for infinite quantum well



$$\hat{H} = \frac{\hat{p}^2}{2m}$$

From the ansatz for $\hat{\eta}_j$

$$\frac{1}{2m} f_1^2 + \frac{\hbar}{2m}\frac{df_1}{dx} + E_1 = 0$$

$$\boxed{\int \frac{dx}{x^2+a} = \frac{1}{\sqrt{a}}\arctan\frac{x}{\sqrt{a}} \quad a>0}$$

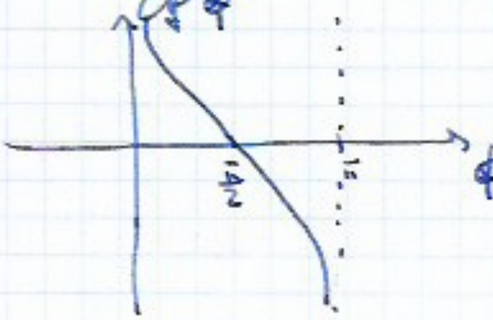
$$\hbar \frac{df_1}{dx} = -\left(f_1^2 + 2mE_1\right)$$

$$\hbar \int \frac{df_1}{f_1^2 + 2mE_1} = -\int dx$$

$$\frac{\hbar}{\sqrt{2mE_1}} \arctan\left(\frac{f_1}{\sqrt{2mE_1}}\right) = -x + b$$

$$f_1(x) = \sqrt{2mE_1}\tan\left(\frac{\sqrt{2mE_1}}{\hbar}(b-x)\right) =$$

$$= \sqrt{2mE_1}\cot\left[\frac{\sqrt{2mE_1}}{\hbar}(x-a)\right]$$



a – arbitrary integration constant

We take $a=0$, and $\frac{\sqrt{2mE_1}}{\hbar}L = \pi$ – ctg function is finite in $(0, L)$

Thus

$$\boxed{\begin{aligned} E_1 &= \frac{\pi^2\hbar^2}{2mL^2} \\ f_1(x) &= \frac{\pi\hbar}{L}\,\mathrm{ctg}\left(\frac{\pi x}{L}\right)\end{aligned}}$$

$$\boxed{\hat{\eta}_1 = \frac{1}{\sqrt{2m}}\left(\hat{p} + \frac{i\pi\hbar}{L}\,\mathrm{ctg}\left(\frac{\pi x}{L}\right)\right)}$$

Next, we need to find f_2, f_3, etc.

We make a short cut with assumption that

$$\hat{\eta}_j = \frac{1}{\sqrt{2m}} \left(\hat{p} + i\, c_j \cot(b_j x) \right)$$

$$c_j, b_j \in \mathbb{R}$$

and $0 \le b_j \le \pi/L$ because in $0 < x < L$ cot must be finite.

Compute

$$\hat{\eta}_j^+ \hat{\eta}_j = \frac{1}{2m} \left(\hat{p}^2 - c_j b_j \hbar + c_j (c_j - b_j \hbar) \cot^2(b_j x) \right)$$

$$\hat{\eta}_j \hat{\eta}_j^+ = \frac{1}{2m} \left(\hat{p}^2 + c_j b_j \hbar + c_j (c_j + b_j \hbar) \cot^2(b_j x) \right)$$

$$(\operatorname{ctg} x)' = -\frac{1}{\sin^2 x} = -(1 + \operatorname{ctg}^2 x)$$

Hence, the recursive relation $(**)$ becomes

$$\frac{1}{2m} \left(\cancel{p^2} - c_{j+1} b_{j+1} \hbar + c_{j+1} (c_{j+1} - b_{j+1} \hbar) \cot^2(\underline{b_{j+1} x}) \right) + E_{j+1} =$$

$$= \frac{1}{2m} \left(\cancel{p^2} + c_j b_j \hbar + c_j (c_j + b_j \hbar) \cot^2(\underline{b_j x}) \right) + E_j$$

compare coefficients of different powers of x

$$\Rightarrow \begin{cases} b_{j+1} = b_j \\ c_{j+1} (c_{j+1} - b_{j+1} \hbar) = c_j (c_j + b_j \hbar) \\ 2m E_{j+1} - c_{j+1} b_{j+1} \hbar = 2m E_j + c_j b_j \hbar \end{cases}$$

$\Rightarrow$ ~~$2m E_{j+1} - c_{j+1} b_{j+1} \hbar = 2m E_j + c_j b_j \hbar$~~

$$2m E_{j+1} - (c_{j+1})^2 = 2m E_j - (c_j)^2$$

or $\quad 2mE_j - c_j^2 = 2mE_1 - c_1^2 = 0 \quad$ from recursion relation

Hence

$$\boxed{E_j = \frac{c_j^2}{2m}}$$

Using $\quad k_j = k_1 = \frac{\pi}{L} \quad$ we get

$$c_{j+1}\left(c_{j+1} - \frac{\pi\hbar}{L}\right) = c_j\left(c_j + \frac{\pi\hbar}{L}\right)$$

There are two solutions $\quad c_{j+1} = -c_j \quad$ or

$$c_{j+1} = c_j + \pi\hbar/L$$ ← this gives larger value of E_{j+1}

hence

$$c_j = j\frac{\pi\hbar}{L}$$

and

$$\boxed{E_j = \frac{\pi^2\hbar^2}{2mL^2} j^2}$$

Next we need to show that there exists a solution to $\hat{\eta}_j |\xi_j\rangle = 0$

In position representation $\quad \xi_j(x) = \langle x|\xi_j\rangle$

$$\left(\frac{\hbar}{i}\frac{d}{dx} + i\frac{j\pi\hbar}{L}\cot\frac{\pi x}{L}\right)\xi_j(x) = 0$$

$$\frac{d\xi_j(x)}{dx} = j\frac{\pi}{L}\cot\left(\frac{\pi x}{L}\right)\xi_j(x)$$

$\int \cot ax\,dx = \frac{1}{a}\ln|\sin ax|$

$$\int \frac{1}{\xi_j} d\xi_j = j\frac{\pi}{L}\int \cot\left(\frac{\pi x}{L}\right)dx$$

$$\ln\xi_j = j\frac{\pi}{L}\cdot\frac{L}{\pi}\ln\left|\sin\left(\frac{\pi x}{L}\right)\right| + C$$

$$\boxed{\xi_j(x) = N_j \sin\left(\frac{\pi x}{L}\right)^j}$$

Therefore,

$$\psi_1(x) = \xi_1(x) = N_1 \sin\left(\frac{\pi x}{L}\right) \qquad N_1 = \sqrt{\frac{2}{L}}$$

$$\psi_2(x) = \hat{\eta}_1^+ \xi_2(x) = N_2\left(\frac{\hbar}{i}\frac{d}{dx} - \frac{i\hbar\pi}{L}\cot\left(\frac{\pi x}{L}\right)\right)\sin^2\frac{\pi x}{L} = \sqrt{\frac{2}{L}}\sin\left(\frac{2\pi x}{L}\right)$$

$$\psi_3(x) = \hat{\eta}_1^+ \hat{\eta}_2^+ \xi_3(x) =$$

$$= \left(\frac{\hbar}{i}\frac{d}{dx} - \frac{i\hbar\pi}{L}\cot\frac{\pi x}{L}\right)\left(\frac{\hbar}{i}\frac{d}{dx} - \frac{2i\hbar\pi}{L}\cot\frac{\pi x}{L}\right)\sin^3\frac{\pi x}{L} =$$

$$= \sqrt{\frac{2}{L}}\sin\left(\frac{3\pi x}{L}\right)$$

$\vdots$

$\square$