Gaussian averages and Wick's theorem

$$\langle F(\bar{x})\rangle \equiv \mathcal{N}(\mathbb{A})\int d^n x\, F(\bar{x})\, e^{-\mathbb{A}(\bar{x})}$$

$\uparrow$ normalization $\langle 1\rangle = 1$

$$\mathcal{N}(\mathbb{A}) = Z(\mathbb{A},0)^{-1} = (2\pi)^{-n/2}(\det \mathbb{A})^{1/2}$$

Thus the generating function

$$\langle e^{\bar{b}\cdot\bar{x}}\rangle = \frac{Z(\mathbb{A},\bar{b})}{Z(\mathbb{A},0)}$$

$$\boxed{\langle x_{k_1} x_{k_2} \cdots x_{k_l}\rangle = (2\pi)^{-n/2}(\det \mathbb{A})^{1/2}\left(\frac{\partial}{\partial b_{k_1}}\cdots\frac{\partial}{\partial b_{k_l}} Z(\mathbb{A},\bar{b})\right)\Big|_{\bar{b}=0}} =$$

$$\boxed{Z(\mathbb{A},\bar{b}) = (2\pi)^{n/2}(\det \mathbb{A})^{-1/2} e^{\Delta(\bar{b})}}$$

$$= \boxed{\left(\frac{\partial}{\partial b_{k_1}}\cdots\frac{\partial}{\partial b_{k_l}} e^{\Delta(\bar{b})}\right)\Big|_{\bar{b}=0}}$$

For general $F(\bar{x})$

$$\boxed{\langle F(\bar{x})\rangle = \left(F\left(\frac{\partial}{\partial \bar{b}}\right) e^{\Delta(\bar{b})}\right)\Big|_{\bar{b}=0}}$$

Wick's theorem

observe $\quad \frac{\partial}{\partial b_k} e^{\Delta(\bar{b})} = \sum_{k'} \Delta_{kk'} b_{k'} e^{\Delta(\bar{b})}$

each derivative generates b_k

another $\frac{\partial}{\partial b_k}$ must act on the same b_k to have finite result at $\bar{b}=0$

to get $\langle x_{k_1} \cdots x_{k_l} \rangle$ with a Gaussian measure we need to take all possible pairings of indices $k_1 \ldots k_l$ (l must be even). Each $k_p k_q$ pair is associated with $\Delta_{k_p k_q}$

Then

$$\boxed{\begin{aligned} \langle x_{k_1} \cdots x_{k_l} \rangle &= \sum_{\substack{\text{all pairings} \\ P \text{ of } \{k_1, \ldots k_l\}}} \Delta_{k_{P_1} k_{P_2}} \cdots \Delta_{k_{P_{l-1}} k_{P_l}} = \\ &= \sum_{\substack{\text{all pairings} \\ P \text{ of } \{k_1, \ldots k_l\}}} \langle x_{k_{P_1}} x_{k_{P_2}} \rangle \cdots \langle x_{k_{P_{l-1}}} x_{k_{P_l}} \rangle \end{aligned}}$$

Examples

$$\langle x_{i_1} x_{i_2} \rangle = \Delta_{i_1 i_2}$$

$$\langle x_{i_1} x_{i_2} x_{i_3} x_{i_4} \rangle = \Delta_{i_1 i_2} \Delta_{i_3 i_4} + \Delta_{i_1 i_3} \Delta_{i_2 i_4} + \Delta_{i_1 i_4} \Delta_{i_3 i_2}$$

in general

$$\langle \underbrace{x \cdots x}_{2p \text{ variables}} \rangle = \underbrace{\overbrace{\Delta \Delta \cdots \Delta}^{p} + \overbrace{\Delta \Delta \Delta \cdots \Delta}^{p} + \cdots}_{(2p-1)(2p-3)\cdots 5 \cdot 3 \cdot 1 \text{ terms}}$$