Gaussian integrals in $\mathbb{R}^n$

$$\boxed{Z(A) = \int d^n x \, e^{-A(\bar{x})}} \qquad A(\bar{x}) = \frac{1}{2}\sum_{ij} x_i A_{ij} x_j$$

The integral converges for the real symmetric matrix A, which is strictly positive (all eigenvalues are positive)

Then $$\boxed{Z(A) = (2\pi)^{n/2} (\det A)^{-1/2}}$$

This is generalization of $\int_{-\infty}^{\infty} dx \, e^{-\frac{ax^2}{2}} = \sqrt{\frac{2\pi}{a}}$ for $n=1$.

<u>proof</u> real symmetric matrix can be diagonalized

$$A = O D O^T \qquad O \text{ - orthogonal matrix}$$

D - diagonal matrix $\qquad O^T O = 1$

$$D_{ij} = a_i \delta_{ij} \qquad a_i > 0$$

changing variables $\quad x_i = \sum_{j=1}^{n} O_{ij} y_j$

$$\sum_{ij} x_i A_{ij} x_j = \sum_{ijk} x_i O_{ik} a_k O_{kj} x_j = \sum_i a_i y_i^2$$

Jacobian of the transformation $|\det O| = 1$. Hence, the integral factorizes

$$Z(A) = \prod_{i=1}^{n} \int dy_i \, e^{-\frac{a_i y_i^2}{2}} = (2\pi)^{n/2} (a_1 a_2 \dots a_n)^{-1/2} =$$
$$= (2\pi)^{n/2} (\det A)^{-1/2} = (2\pi)^{n/2} \left(e^{\operatorname{Tr} \ln A}\right)^{-1/2}$$

$$\det A = a_1 \dots a_n = e^{\ln a_1 \dots a_n} = e^{\sum \ln a_i} = e^{\operatorname{Tr} \ln A} \qquad \blacksquare$$

In case of complex symmetric matrices $A = U D U^T$, where U is unitary matrix and D diagonal positive matrix