Generating function for cumulants

Let

$$Z(\bar{b},\lambda) = \int d^n x \, e^{-A(\bar{x},\lambda) + \bar{b}\cdot\bar{x}}$$

and

$$\langle e^{\bar{b}\cdot\bar{x}} \rangle_\lambda = \frac{Z(\bar{b},\lambda)}{Z(\lambda)}$$

and we find

$$\boxed{\langle x_{i_1} x_{i_2} \cdots x_{i_l} \rangle_\lambda = \frac{1}{Z(\lambda)} \left[\frac{\partial}{\partial b_{i_1}} \cdot \frac{\partial}{\partial b_{i_2}} \cdot \cdots \frac{\partial}{\partial b_{i_l}} Z(\bar{b},\lambda) \right] \Big|_{\bar{b}=0}}$$

Generating function for cumulants – connected l-point correlation functions

$$\boxed{\begin{aligned} W(\bar{b},\lambda) &= \ln Z(\bar{b},\lambda) \\ W^{(l)}_{i_1 i_2 \cdots i_l} &= \left[\frac{\partial}{\partial b_{i_1}} \cdot \frac{\partial}{\partial b_{i_2}} \cdots \frac{\partial}{\partial b_{i_l}} W(\bar{b},\lambda) \right] \Big|_{\bar{b}=0} \end{aligned}}$$

Example

$$V(\bar{x}) = \frac{1}{4!} \sum_{i=1}^{n} x_i^4$$

- $W_i^{(1)} = \langle x_i \rangle_\lambda$
- $W^{(2)}_{i_1 i_2} = \langle x_{i_1} x_{i_2} \rangle_\lambda - \langle x_{i_1} \rangle_\lambda \langle x_{i_2} \rangle_\lambda = \langle (x_{i_1} - \langle x_{i_1} \rangle_\lambda)(x_{i_2} - \langle x_{i_2} \rangle_\lambda) \rangle_\lambda$
- if $V(\bar{x}) = V(-\bar{x})$

$$W^{(2)}_{i_1 i_2} = \langle x_{i_1} x_{i_2} \rangle_\lambda$$

- $W^{(4)}_{i_1 i_2 i_3 i_4} = \langle x_{i_1} x_{i_2} x_{i_3} x_{i_4} \rangle_\lambda - \langle x_{i_1} x_{i_2} \rangle_\lambda \langle x_{i_3} x_{i_4} \rangle_\lambda -$

$$- \langle x_{i_1} x_{i_3} \rangle_\lambda \langle x_{i_2} x_{i_4} \rangle_\lambda - \langle x_{i_1} x_{i_4} \rangle_\lambda \langle x_{i_2} x_{i_3} \rangle_\lambda$$

For an x^4 perturbation

$$W^{(4)}_{i_1 i_2 i_3 i_4} = -\lambda \sum_i \Delta_{i_1 i}\Delta_{i_2 i}\Delta_{i_3 i}\Delta_{i_4 i} +$$

$$+\frac{1}{2}\lambda^2 \sum_{ij} \Delta_{i_1 i}\Delta_{i_2 i}\Delta_{i_3 j}\Delta_{i_4 j}\Delta_{ij}^2 +$$

$$+\frac{1}{2}\lambda^2 \sum_{ij} \Delta_{i_1 i}\Delta_{i_3 i}\Delta_{i_2 j}\Delta_{i_4 j}\Delta_{ij}^2 +$$

$$+\frac{1}{2}\lambda^2 \sum_{ij} \Delta_{i_1 i}\Delta_{i_4 i}\Delta_{i_3 i}\Delta_{i_2 j}\Delta_{ij}^2 +$$

$$+\frac{1}{2}\lambda^2 \sum_{ij} \left(\Delta_{ii}\Delta_{jj}\Delta_{i_1 i}\Delta_{i_2 j}\Delta_{i_3 i}\Delta_{i_4 i} + 3 \text{ terms}\right) + \mathcal{O}(\lambda)$$

Diagramatic representation