Generating function

$\Omega(x_1, x_2 \dots x_n)$ ~ positive, normalized measure $\mathbb{R}^n$
probability distribution

average value of $F(\bar{x}) = F(x_1, \dots x_n)$

$$\langle F \rangle = \int d^n x \, F(\bar{x}) \, \Omega(\bar{x}) \qquad d^n x = \prod_{i=1}^{n} dx_i$$

of course $\langle 1 \rangle = 1$

<u>Fact:</u> Fourier transform of a distribution is a generating function of its moments.

$$(*) \quad \boxed{Z(\bar{b}) = \langle e^{\bar{b}\cdot\bar{x}} \rangle = \int d^n x \, \Omega(\bar{x}) e^{\bar{b}\cdot\bar{x}}} \qquad \bar{b} \in \mathbb{R}^n$$

$$\bar{b}\cdot\bar{x} = \sum_{i=1}^{n} b_i x_i$$

Expanding in powers of b_k

$$\boxed{Z(\bar{b}) = \sum_{l=0}^{\infty} \frac{1}{l!} \sum_{k_1 \dots k_l = 1}^{n} b_{k_1} b_{k_2} \dots b_{k_l} \langle x_{k_1} x_{k_2} \dots x_{k_l} \rangle}$$

Differentiating both sides of $*$)

$$\frac{\partial}{\partial b_k} Z(\bar{b}) = \int d^n x \, x_k \, e^{\bar{b}\cdot\bar{x}} \, \Omega(\bar{x})$$

Differentiating many times and taking $\bar{b} = 0$

$$\boxed{\langle x_{k_1} x_{k_2} \dots x_{k_l} \rangle = \frac{\partial}{\partial b_{k_1}} \frac{\partial}{\partial b_{k_2}} \cdots \frac{\partial}{\partial b_{k_l}} Z(\bar{b}) \Big|_{\bar{b}=0}}$$