HARMONIC OSCILLATOR

classical

$$m \frac{d^2 x}{dt^2} = -kx$$

$$\omega_0^2 = \frac{k}{m} \quad \left[\frac{1}{s}\right] \quad \text{frequency}$$

$$\frac{d^2 x}{dt^2} + \omega_0^2 x = 0 \;\Big|\; \dot{x} \qquad \rightarrow x(t) = x_0 \sin(\omega_0 t + \varphi)$$

$$\frac{d}{dt}\left\{\frac{1}{2}\left(\dot{x}^2 + \omega_0^2 x^2\right)\right\} = 0 \rightarrow \frac{E}{m} = \frac{1}{2}\left(\dot{x}^2 + \omega_0^2 x^2\right) = \text{const.}$$

$$E = \underbrace{\frac{1}{2} m \dot{x}^2}_{E_{kin}} + \underbrace{\frac{k}{2} x^2}_{E_{pot.}}$$

quantum

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{k}{2}\hat{x}^2 \qquad \beta^2 = \frac{m\omega_0}{\hbar}$$

$$\hat{H}|\psi\rangle = E|\psi\rangle \qquad \xi = \beta x$$

dimensionless length

Algebraic way to obtain spectrum

$$\boxed{\begin{aligned}\hat{a} &= \frac{\beta}{\sqrt{2}}\left(\hat{x} + \frac{i}{m\omega_0}\hat{p}\right)\\ \hat{a}^+ &= \frac{\beta}{\sqrt{2}}\left(\hat{x} - \frac{i}{m\omega_0}\hat{p}\right)\end{aligned}} \qquad \hat{a} \neq \hat{a}^+$$

$$[\hat{x}, \hat{p}] = i\hbar \rightarrow [\hat{a}, \hat{a}^+] = 1 \qquad \hat{a}\hat{a}^+ = 1 + \hat{a}^+\hat{a}$$

Prof. $[\hat{a}, \hat{a}^+] = \hat{a}\hat{a}^+ - \hat{a}^+\hat{a} = \frac{\beta^2}{2}\left\{\left(\hat{x} + \frac{i\hat{p}}{m\omega_0}\right)\left(\hat{x} - \frac{i\hat{p}}{m\omega_0}\right) -\right.$

$\left. - \left(\hat{x} - \frac{i\hat{p}}{m\omega_0}\right)\left(\hat{x} + \frac{i\hat{p}}{m\omega_0}\right)\right\} = 2\frac{\beta^2}{2}\left(-i\frac{\hat{x}\hat{p}}{m\omega_0} + i\frac{\hat{p}\hat{x}}{m\omega_0}\right) =$

$= \beta^2\left(-\frac{i}{m\omega_0}\right)(\hat{x}\hat{p} - \hat{p}\hat{x}) = \beta^2\left(-\frac{i}{m\omega_0}\right)i\hbar = \frac{m\omega_0}{\hbar}\cdot\frac{\hbar}{m\omega_0} = 1$ $\square$

$$\boxed{\begin{aligned}\hat{x} &= \frac{\hat{a}+\hat{a}^{+}}{\sqrt{2}\,\beta}\\ \hat{p} &= \frac{m\omega_0}{i}\,\frac{\hat{a}-\hat{a}^{+}}{\sqrt{2}\,\beta}\end{aligned}}$$

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{k}{2}\hat{x}^2 = \hbar\omega_0\left(\hat{a}^{+}\hat{a} + \frac{1}{2}\right)$$

<u>Proof</u>.

$$\hat{H} = \frac{1}{2m}\left(-\frac{m^2\omega_0^2}{2\beta^2}\right)(\hat{a}-\hat{a}^{+})^2 + \underbrace{\frac{k}{2}}_{=\omega_0^2 m}\frac{1}{2\beta^2}(\hat{a}+\hat{a}^{+})^2 =$$

$$= -\frac{\omega_0^2 m}{4\beta^2}\left(\hat{a}^2 + \hat{a}^{+2} - \hat{a}\hat{a}^{+} - \hat{a}^{+}\hat{a}\right) + \frac{k}{4\beta^2}\left(\hat{a}^2 + \hat{a}^{+2} + \hat{a}^{+}\hat{a} + \hat{a}\hat{a}^{+}\right) =$$

$$= \frac{\omega_0^2 m}{4\beta^2}\left(-\hat{a}^2 - \hat{a}^{+2} + \hat{a}\hat{a}^{+} + \hat{a}^{+}\hat{a} + \hat{a}^2 + \hat{a}^{+2} + \hat{a}\hat{a}^{+} + \hat{a}^{+}\hat{a}\right) =$$

$$= \frac{\omega_0^2 m}{2\beta^2}\left(\hat{a}^{+}\hat{a} + \hat{a}\hat{a}^{+}\right) = \frac{\omega_0^2 \cancel{m}}{2\cancel{m\omega_0}}\hbar\left(2\hat{a}^{+}\hat{a} + 1\right) =$$

$$= \hbar\omega_0\left(\hat{a}^{+}\hat{a} + \frac{1}{2}\right)$$

$\square$

Eigen value of $\hat{H}$ $\leftrightarrow$ Eigen value of $\boxed{\hat{N} = \hat{a}^{+}\hat{a}}$

particle number operator

$$\hat{N}|n\rangle = \hat{a}^{+}\hat{a}|n\rangle = n|n\rangle$$

$\hat{N}^{+} = \hat{N}$ – hermitian $\rightarrow$ $\underline{n \in \mathbb{R}}$

$$\hat{a}\hat{a}^{+} - \hat{a}^{+}\hat{a} = 1$$

•) $\hat{N}\,\hat{a}\,|n\rangle = \underbrace{\hat{a}^{+}\hat{a}}_{\hat{a}\hat{a}^{+}-1}\,\hat{a}\,|n\rangle = (\hat{a}\hat{a}^{+}-1)\hat{a}\,|n\rangle =$

$= \hat{a}\,(\hat{a}^{+}\hat{a} - 1)\,|n\rangle = \hat{a}\,(\hat{N} - 1)\,|n\rangle =$

$= \hat{a}\,(n-1)\,|n\rangle = (n-1)\,\underbrace{\hat{a}\,|n\rangle}_{\text{eigenstate of } \hat{N} \text{ with } n-1}$

Hence

$$\hat{a}\,|n\rangle = \alpha_n\,|n-1\rangle$$
$$\hat{a}\,|n-1\rangle = \alpha_{n-1}\,|n-2\rangle$$

etc.

$\hat{a}$ – annihilation operator → lowering n by one.

•) $\hat{N}\,\hat{a}^{+}\,|n\rangle = \hat{a}^{+}\hat{a}\,\hat{a}^{+}\,|n\rangle = \hat{a}^{+}\,(\hat{a}^{+}\hat{a} + 1)\,|n\rangle =$

$= \hat{a}^{+}\,(\hat{N} + 1)\,|n\rangle = \hat{a}^{+}\,(n+1)\,|n\rangle =$

$= (n+1)\,\underbrace{\hat{a}^{+}\,|n\rangle}_{\text{eigenstate of } \hat{N} \text{ with } n+1}$

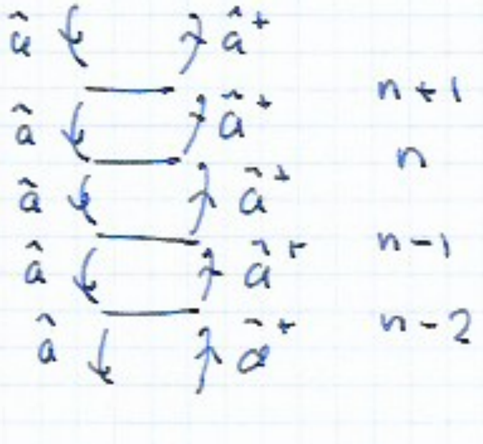
Hence

$$\hat{a}^{+}\,|n\rangle = \tilde{\alpha}_n\,|n+1\rangle$$
$$\hat{a}^{+}\,|n+1\rangle = \tilde{\alpha}_{n+1}\,|n+2\rangle$$

etc.

$\hat{a}^{+}$ – creation operator → increases n by one

<u>Ladder operators</u>



③

If $\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2} m \omega_0^2 \hat{x}^2$ - sum of squares of two hermitian operators

then $\langle \hat{H} \rangle \geq 0$.

<u>Proof</u> $\hat{A}^+ = \hat{A}$

$$\langle \hat{A}^2 \rangle = \int \psi^* \hat{A}^2 \psi = \int (\hat{A}\psi)^* (\hat{A}\psi) = \int |\hat{A}\psi|^2 \geq 0$$

$\square$

Hence $\hat{H} | n) = \hbar \omega (\hat{N} + \frac{1}{2}) | n) = \hbar \omega_0 (n + \frac{1}{2}) | n)$

$$\langle n | \hat{H} | n \rangle = \hbar \omega_0 (n + \tfrac{1}{2}) \geq 0$$

$\Rightarrow$ $\underline{n \geq -\frac{1}{2}}$ there are no states with $n < -\frac{1}{2}$.

There exists n^* such that $\hat{a} | n^* \rangle = 0$

since $\hat{a} | n^* \rangle = | n^* - 1 \rangle = 0$

$\hat{a} | n^* - 1 \rangle = | n^* - 2) = 0$

etc.

$\hat{N} | n \rangle = n | n)$

We also have $\hat{N} | n^* \rangle = \hat{a}^+ \underbrace{\hat{a} | n^* \rangle}_{=0} = 0 | n^*)$

$\Rightarrow$ $\underline{n^* = 0}$

vaccum state $\boxed{\hat{a} | 0 \rangle = 0}$

$$\hat{N}\,\hat{a}^+|0\rangle = \hat{a}^+\hat{a}\,\hat{a}^+|0\rangle =$$

$$= \hat{a}^+(\hat{a}^+\hat{a}+1)|0\rangle = \underbrace{\;}_{=\mathbb{1}}\,\underbrace{\hat{a}^+|0\rangle}_{|1\rangle} = 1\,|1\rangle$$

etc. $\quad \hat{N}\,\underbrace{\hat{a}^+|n\rangle}_{|n+1\rangle} = (n+1)\,\underbrace{\hat{a}^+|n\rangle}_{|n+1\rangle}$

$\Rightarrow \quad n \in \mathbb{Z} \qquad n = 0, 1, 2, 3, \ldots$

and $\quad \hbar\omega_0 (n+\frac{1}{2}) \geq 0$

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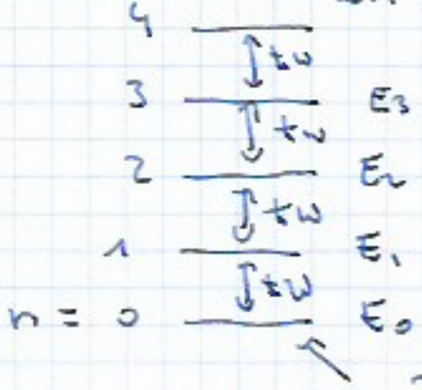
$$\hat{H}|n\rangle = \hbar\omega_0 (n+\tfrac{1}{2})|n\rangle$$

$$E_n = \hbar\omega_0(n+\tfrac{1}{2}) \qquad n = 0, 1, 2, \ldots$$



zero mode, groundstate

particle number ~~operator~~
representation

or

energy representation