Gas of ideal fermions at $T=0$ in 3 dimensions

$$\hat H = \sum_\sigma \int d_3 r\; \hat\psi^\dagger_\sigma(\vec r)\left[-\frac{\hbar^2}{2m}\nabla^2\right]\hat\psi_\sigma(\vec r) =$$

$$= \sum_{\vec k\sigma} \underbrace{\frac{\hbar^2 k^2}{2m}}_{\varepsilon_{\vec k}} \hat a^\dagger_{\vec k\sigma}\hat a_{\vec k\sigma} = \sum_{\vec k\sigma} \varepsilon_{\vec k}\, \hat n_{\vec k\sigma}$$

$\{|\vec k\sigma\rangle\}$ - base $\quad \langle \vec r\sigma'|\vec k\sigma\rangle = \frac{1}{\sqrt V} e^{i\vec k\cdot\vec r}\,\delta_{\sigma\sigma'}$

P.B.C. $\quad \vec k = \frac{2\pi}{L}\vec n$

$\vec n = (n_x, n_y, n_z) \quad n_i \in \mathbb{Z}$

$$\Psi(x+L,y,z) = \Psi(x,y,z)$$

$$e^{ik_x(x+L)} = e^{ik_x x} \Rightarrow e^{ik_x L} = 1 = e^{2\pi n i}$$

□

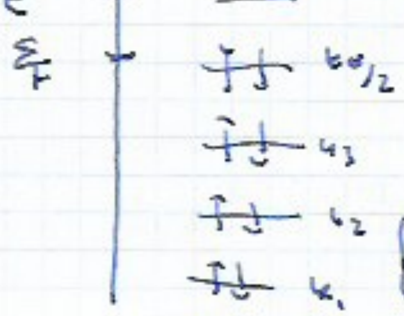
Some order of one particle base

$$|k_1\uparrow\rangle, |k_1\downarrow\rangle, |k_2\uparrow\rangle, |k_2\downarrow\rangle, \dots$$

$$\varepsilon_{k_1} \le \varepsilon_{k_2} \le \varepsilon_{k_3} \le \dots$$

Ground state $\quad T=0$

$$\boxed{|FS\rangle = a^\dagger_{k_1\uparrow} a^\dagger_{k_1\downarrow} a^\dagger_{k_2\uparrow} a^\dagger_{k_2\downarrow} \dots a^\dagger_{k_{N/2}\uparrow} a^\dagger_{k_{N/2}\downarrow} |vac\rangle}$$



$\boxed{p_F = \hbar k_F}$ Fermi momentum

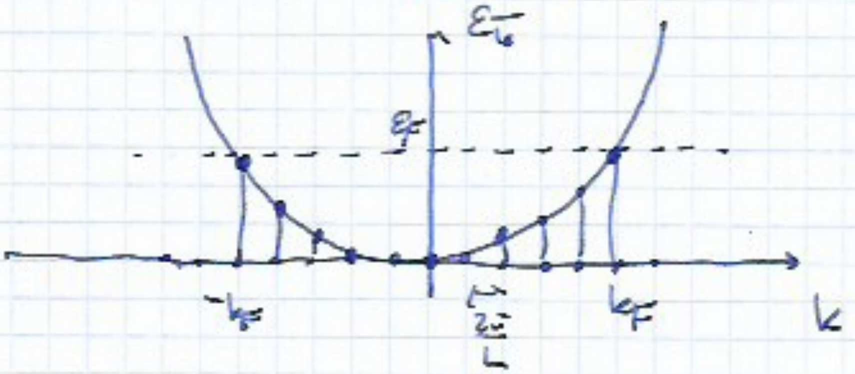
$\boxed{\varepsilon_F = \frac{\hbar^2 k_F^2}{2m}}$ Fermi energy

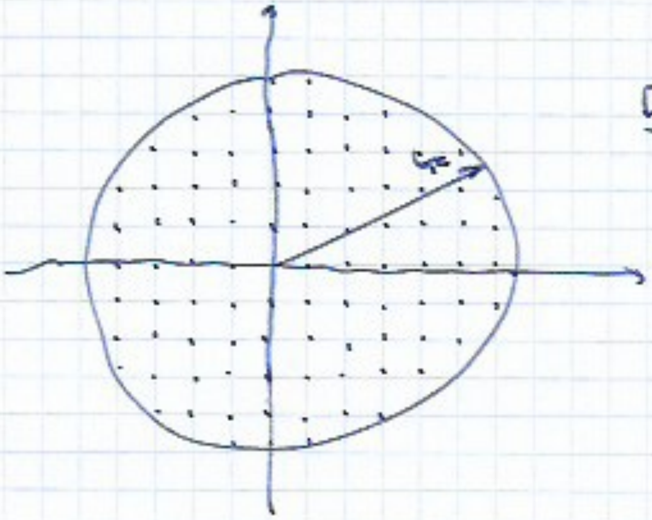
$\boxed{k_F = \frac{1}{\hbar}\sqrt{2m\varepsilon_F}}$ Fermi wave vector

$\boxed{\lambda_F = \frac{2\pi}{k_F}}$ Fermi wave length

$\boxed{T_F = \frac{\varepsilon_F}{k_B}}$ Fermi temperature

$\boxed{v_F = \frac{\hbar k_F}{m}}$ Fermi velocity





Fermi sea

$$\hat n_{k\sigma}|FS\rangle = \begin{cases} 1\,|FS\rangle & |k|\le k_F \\ 0\,|FS\rangle & |k|>k_F \end{cases}$$

$\Theta(k_F-|\vec k|)$ – step function

Thermodynamic limit $\quad N\to\infty,\ V\to\infty,\ \frac{N}{V}=n=\text{const.}$

$$N=\langle FS|\hat N|FS\rangle = \langle FS|\sum_{k\sigma}\hat n_{k\sigma}|FS\rangle =$$

$$=\sum_{k\sigma}\langle FS|\hat n_{k\sigma}|FS\rangle = \sum_{k\sigma}\Theta(k_F-|\vec k|)\overbrace{\langle FS|FS\rangle}^{=1}$$

$$=\underbrace{2}_{\text{spin}}\frac{L^3}{(2\pi)^3}\int d^3k\ \Theta(k_F-|\vec k|) = \quad \text{spherical coordinates}$$

$$=2\frac{L^3}{(2\pi)^3}\underbrace{\int_0^{2\pi}d\varphi\int_0^{\pi}d\theta\,\sin\theta}_{4\pi}\ \underbrace{\int_0^{k_F}dk\,k^2}_{\frac{1}{3}k_F^3} = \frac{V}{3\pi^2}k_F^3$$

$$\Rightarrow\quad \boxed{n=\frac{N}{V}=\frac{k_F^3}{3\pi^2}} \longleftrightarrow \boxed{k_F=\left(3\pi^2 n\right)^{1/3}}$$

E.g. | Cu $\qquad n = 8.47\cdot10^{28}\ 1/m^3$

$k_F = 13.6\cdot10^{9}\ 1/m = 13.6\ nm^{-1}$ | $T_F = 10^5\ K$

$\lambda_F = 0.46\ nm$ | $v_F = 10^6\ m/s = 0.5‰\cdot c$

$\varepsilon_F = \frac{\hbar^2k_F^2}{2m} = 10^{-18}\ J = 7\ eV$

Metals

	$n = \frac{N}{V}\left[\frac{1}{cm^3}\right]$	$v_F\left[\frac{cm}{s}\right]$	$E_F\,[eV]$	$T_F\,[K]$
Li	$4.6\cdot10^{22}$	$1.3\cdot10^{8}$	4.7	$5.5\cdot10^{4}$
Na	2.5	1.1	3.1	3.7
K	1.34	0.85	2.1	2.4
Rb	1.08	0.79	1.8	2.1
Cs	0.86	0.73	1.5	1.8
Cu	8.5	1.56	7.0	8.2
Ag	5.76	1.38	5.5	6.4
Au	5.9	1.39	5.5	6.4

$$c = 3\cdot10^{8}\ m/s = 3\cdot10^{10}\ cm/s$$

other systems

matter	particles	$T_F\,[K]$
liquid ^3He	atoms	0.3
metal	electrons	$5\cdot10^{4}$
white dwarfs	electrons	$3\cdot10^{9}$
nuclear matter	nucleons	$3\cdot10^{11}$
neutron stars	neutrons	$3\cdot10^{12}$

As long as $T \ll T_F$ we are in a low-energy regime

$$k_B = 1.38\cdot10^{-23}\ J/K = 8.6\cdot10^{-5}\ \frac{eV}{K}$$

$$1\,eV = 1.6\cdot10^{-19}\ J$$

$$1\,eV \Leftrightarrow 10^{4}\ K$$

Simple estimates

$$n^{-1/3} \sim a_B = \frac{\hbar^2}{me^2} \qquad \text{Bohr's radius}$$

$$k_F \sim \frac{1}{a_B} \qquad \frac{e^2}{\hbar c} \text{ - fine constant}$$

$$\boxed{\alpha = \frac{e^2}{\hbar c} \simeq \frac{1}{137}}$$

$\Rightarrow$

$$\varepsilon_F \sim \frac{\hbar^2}{m a_B^2} \sim \left(\frac{e^2}{\hbar c}\right)^2 mc^2 \sim \left(\frac{1}{137}\right)^2 mc^2$$

$$p_F \sim \frac{\hbar}{a_B} \sim \left(\frac{e^2}{\hbar c}\right) mc \sim \frac{1}{137} mc$$

$$v_F \sim \frac{p_F}{m} \sim \left(\frac{e^2}{\hbar c}\right) c \sim \frac{1}{137} c$$

Internal energy

$$\boxed{E^{(0)}} = \langle FS | \hat{H} | FS \rangle = \sum_{k\sigma} \frac{\hbar^2 k^2}{2m} \langle FS | \hat{n}_{k\sigma} | FS \rangle =$$

$$= 2 \frac{V}{(2\pi)^3} \frac{\hbar^2}{2m} \int d^3k \, k^2 =$$

$$= 2 \frac{V}{(2\pi)^3} \frac{\hbar^2}{2m} 4\pi \int_0^{k_F} dk \, k^4 = \frac{V}{5\pi^2} \frac{\hbar^2}{2m} k_F^5 =$$

$$= \frac{V}{5\pi^2} \varepsilon_F k_F^3 = \boxed{\frac{3}{5} N \varepsilon_F}$$

$$\varepsilon_F = \frac{\hbar^2}{2m} k_F^2 = \frac{\hbar^2}{2m} (3\pi^2 n)^{2/3}$$

$$\boxed{\frac{E^{(0)}}{N} = \frac{3}{5} \varepsilon_F = \frac{3}{5} \frac{\hbar^2}{2m} (3\pi^2)^{2/3} n^{2/3}}$$

or

$$\boxed{E^{(0)} = \frac{3}{5} \frac{\hbar^2}{2m} (3\pi^2)^{2/3} N^{5/3} V^{-2/3}}$$

pressure

$$p = -\left(\frac{\partial E^{(0)}}{\partial V}\right)_N = -\frac{3}{5}\frac{\hbar^2}{2m}(3\pi^2)^{2/3} N^{5/3}\left(-\frac{2}{3}\right) V^{\overbrace{-2/3-1}^{-5/3}} =$$

$$= \frac{2}{5}\frac{\hbar^2}{2m}(3\pi^2)^{2/3} n^{5/3}$$

$1\,Pa = \frac{1N}{1m^2}$

$$\boxed{p = \frac{\hbar^2}{5m}(3\pi^2)^{2/3} n^{5/3}\Big|_{Cu} = 10^6\ atm}$$

$1\,atm = 101325\ Pa$

from sea level pressure

pressure due to Pauli principle

for classical ideal gas

$$p = \frac{N}{V}RT = nRT = \underline{\underline{0}} \quad \text{at } T=0$$

$P > 0$ for electrons yields an instability. One has to add Coulomb interaction between e-e and electrons and ions.

White dwarfs the Pauli pressure is balanced by the gravity attraction stopping a collapse.