FERMI-LIQUID THEORY

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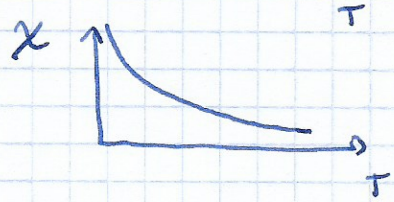
§1. HISTORICAL BACKGROUND

•) classical (Drude) theory ~1920

electrons

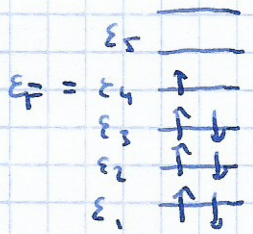
$$C_V = \frac{3}{2} k_B$$

Curie law $\quad \chi = \frac{\mu_B^2}{3 k_B T}$

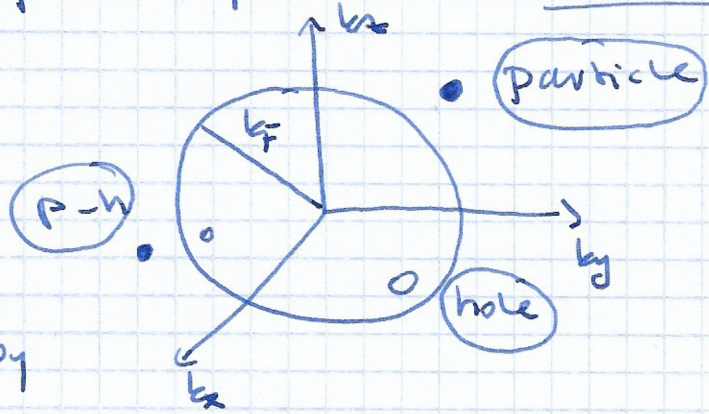


•) Pauli-Sommerfeld model ~1927-28

electrons - exclusion principle



excitations

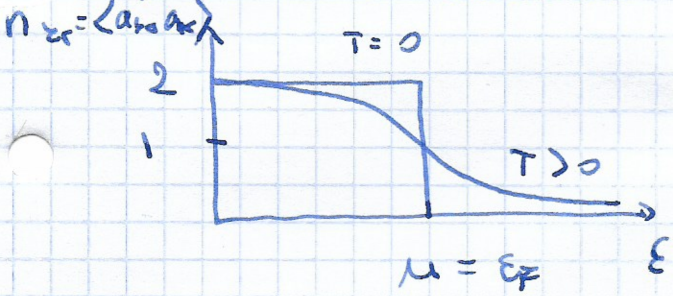


Fermi: level/energy
vector
velocity
wave length

$$\boxed{|GS\rangle = \prod_{k<k_F} \prod_\sigma a^+_{k\sigma} |vac\rangle}$$

elementary excitations

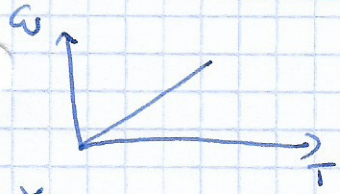
$$\left\{ \begin{aligned} |p\rangle &= a^+_{k>k_F\sigma} |GS\rangle \\ |h\rangle &= a_{k<k_F\sigma} |GS\rangle \\ |ph\rangle &= a^+_{k>k_F\sigma} a_{k<k_F\sigma} |GS\rangle \end{aligned} \right.$$



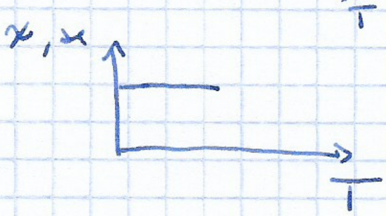
$$\boxed{n_{k\sigma} = \frac{1}{e^{\beta(\varepsilon_k - \mu)} + 1}}$$

Fermi-Dirac

Low-temperature predictions



$c_V \simeq \gamma T$ — specific heat



$\chi \simeq \text{const} + O(T^2)$ — Pauli susceptibility

$\kappa \simeq \text{const} + O(T^2)$ — compressibility

- Interacting electrons in metals and atoms in ^3He

One finds experimentally the same behavior but with renormalized parameters

mass $m \longleftrightarrow m^*$ effective mass

etc.

electron $m_e, e, \sigma \longleftrightarrow$ quasielectron m^*, e, σ

atom ^3He $m_{3He}, \sigma \longleftrightarrow$ quasiatom m^*, σ

$\underbrace{2p + 1n + 2e}_{\text{fermion}}$

nuclear matter, white dwarft, cold fermionic atoms

- Landau - 1957, 58 - Why interacting fermions are so similar to the ideal fermionic gas at low energies or temperatures?

nobel 1962

Our plan
- fenomenological FL and the concept of adiabatic continuity
- microscopic / Green function derivation

- idea of quasiparticles

Landau - Anderson - others

„More is different" P. W. Anderson 1972, Science

emergent properties

complicated many body system	$\longleftrightarrow$	system of weakly interacting quasiparticles

Df. quasiparticle — a particle like object describing elementary excitations of many-body interacting system.

Example: 1) Lattice of interacting ions — phonons

2) oscillating charges — plasmons

3) Landau's Fermi liquid theory — quasielectrons, quasiholes, quasiparticles

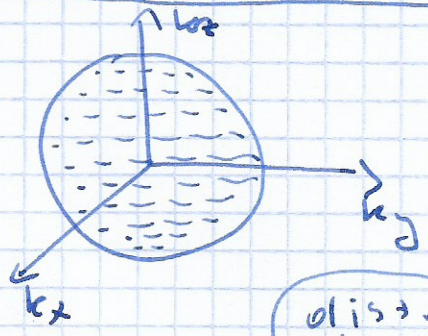
Standard model of interacting fermions

§2. QUASIPARTICLE CONCEPT

•) Ideal Fermi gas

consider $d = 3$ isotropic fermi system

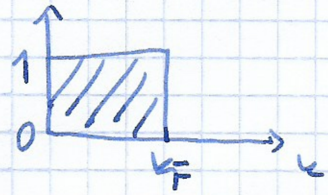
The ground state at $T = 0$



$$|GS\rangle = \prod_{k\sigma}^{k_F} \hat{a}^{+}_{k\sigma} |vac\rangle$$

$$n^0_{k\sigma} = \langle a^{+}_{k\sigma} a_{k\sigma} \rangle = \theta(k_F - |\vec{k}|)$$

distribution function



mean particle density

$$\boxed{n = \frac{1}{V} \sum_{\vec{k}\sigma} \theta(k_F - |\vec{k}|) = 2 \int \frac{d^3k}{(2\pi)^3} \theta(k_F - |\vec{k}|) = \frac{k_F^3}{3\pi^2}}$$

↑ spin $1/2$

ground state energy

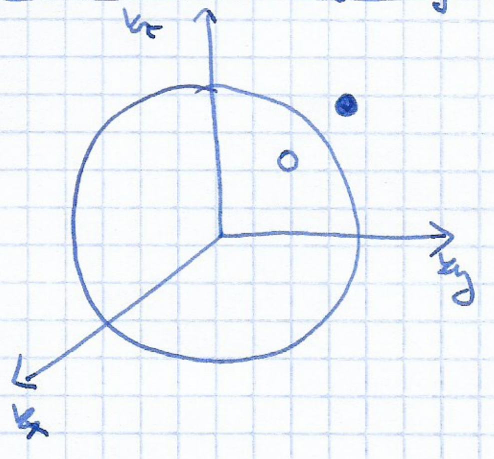
$$\boxed{E_0 = \sum_{k\sigma} \varepsilon^0_{\vec{k}} \, \theta(k_F - |\vec{k}|) = \frac{3}{5} n \varepsilon^0_F V}$$

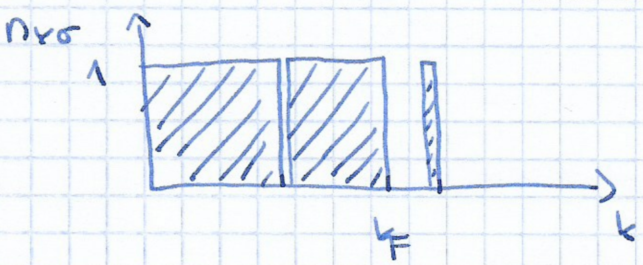
$$\varepsilon^0_{\vec{k}} = \frac{\hbar^2 \vec{k}^2}{2m} \quad - \text{ free fermion dispersion}$$

$\boxed{\hbar = 1}$ Landau

$$\varepsilon^0_F = \frac{\hbar^2 k_F^2}{2m} = \mu\,(T = 0) \quad - \text{ Fermi energy}$$

The low-lying excited states at $T > 0$





$n^0_{k\sigma}$ – at $T = 0$

$n_{k\sigma}$ – at $T > 0$

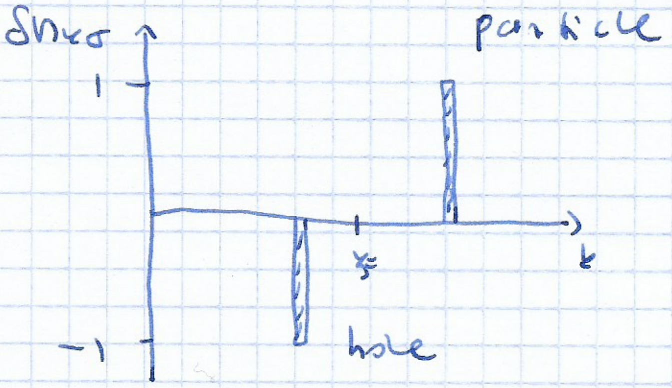
the change in the total energy

$$\boxed{\delta E = E(T>0) - E(T=0)} =$$

$$= \sum_{k\sigma} \varepsilon^0_{k}\, n_{k\sigma} - \sum_{k\sigma} \varepsilon^0_{k}\, n^0_{k\sigma} =$$

$$\boxed{= \sum_{k\sigma} \varepsilon_{k\sigma}\, \delta n_{k\sigma}}$$

$$\boxed{\delta n_{k\sigma} \equiv n_{k\sigma} - n^0_{k\sigma}}$$ – small if $k_B T \ll \varepsilon_F$



Note: The particle energy $\boxed{\varepsilon^0_k = \frac{\delta E[\delta n_{k\sigma}]}{\delta n_{k\sigma}}}$

The particle group velocity $\vec{v}_k = \vec{\nabla}_k\, \varepsilon^0_k$

The Fermi velocity $v_F = |\vec{v}_k|\Big|_{k=k_F} = \frac{\partial \varepsilon^0_k}{\partial |k|}\Big|_{k_F} = \frac{k_F}{m}$

Near the Fermi surface

$$\boxed{\varepsilon^0_k = \varepsilon^0_F + v_F\,(|k| - k_F) + O\left((|k|-k_F)^2\right)}$$

Any excited state is constructed by creating a certain number of particle or hole excitations. Since they are non-interacting the total energy δE is the sum of the particle and hole energies.

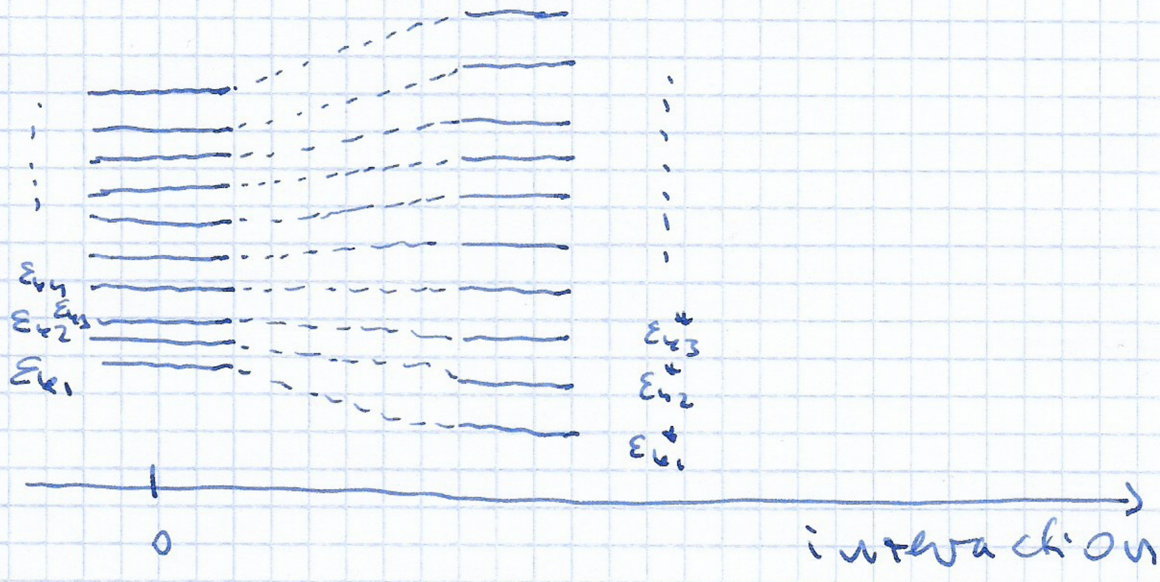
c) Interacting Fermi liquid

Central hypothesis of Landau Fermi-liquid theory:

Any state of the ideal Fermi gas, characterized by a momentum distribution function $\boxed{n_{\vec{k}\sigma} = n^0_{\vec{k}\sigma} + \delta n_{\vec{k}\sigma}}$,

generates an eigenstate of the interacting system as the interaction is switched on adiabatically.

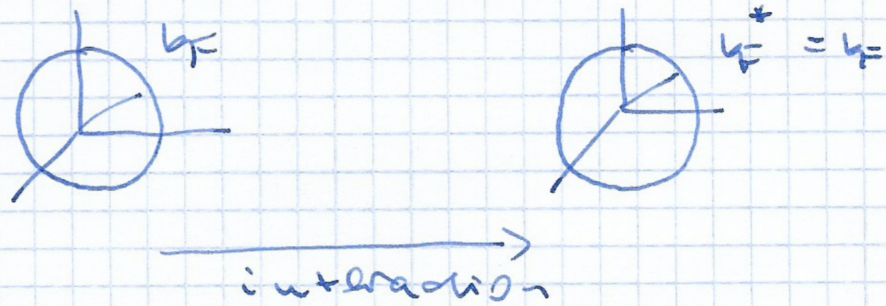
$\boxed{\text{one - to - one correspondence}}$



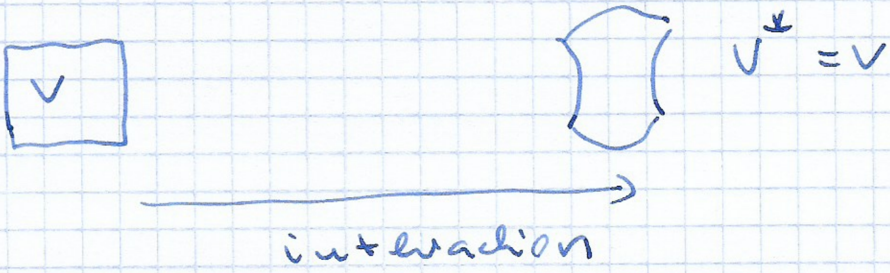
particle		quasiparticle	
electron	$\lvert\vec{k}\rvert > k_F$	quasiparticle	$\lvert\vec{k}\rvert > k_F$
hole	$\lvert\vec{k}\rvert < k_F$	quasihole	$\lvert\vec{k}\rvert < k_F$
charge	$\pm e$	charge	$\pm e$
spin	$\frac{\hbar}{2}$	spin	$\frac{\hbar}{2}$
always stable		stable at low ω	

Eigenstates are labeled by the quasi-particle distribution function $n_{\vec{k}\sigma}$

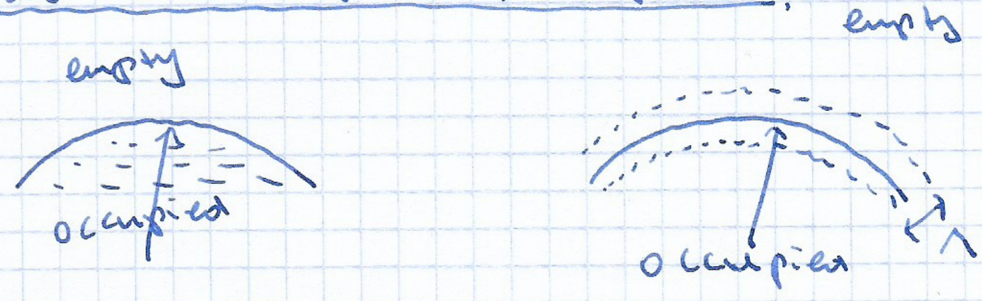
For isotropic system $\mathbb{R}^3$ the Fermi surface is spherical and its volume (and shape) is independent on the interaction (Luttinger theorem)



For anisotropic system only the volume is preserved



near the Fermi surface:



quasiparticle dispersion

$$\varepsilon_{\vec{k}} = \varepsilon_{k_F} + v_F^* (|\vec{k}| - k_F) + O((|\vec{k}| - k_F)^2)$$

$$v_F^* = \frac{k_F}{m^*}$$

effective mass

Fermi velocity

quasiparticle group velocity $\vec{v}_{\vec{k}}^{\,*} = \nabla_{\vec{k}} \varepsilon_{\vec{k}} \rightarrow v_F^* \hat{k}$ for $|\vec{k}| = k_F$

quasiparticle DOS

$$N^*(\xi) = \frac{1}{V} \sum_{\vec{k}} \delta(\xi - \varepsilon_{\vec{k}})$$

$$N^*(0) = \frac{m^* k_F}{2\pi^2}$$

•) <u>Adiabatic switching on the interaction</u>

quasiparticle / quasi hole life time must obey

$$\boxed{\frac{1}{\tau_{\vec{k}}} \ll |\xi_{\vec{k}}| = |\varepsilon_{\vec{k}} - \mu|}$$

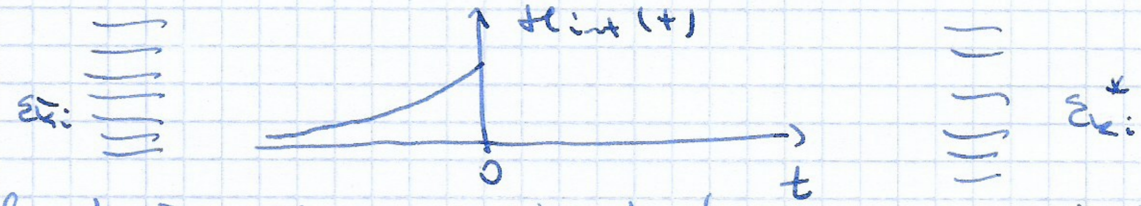
because $\frac{1}{|\xi_{\vec{k}}|}$ is the minimal time to observe / create a quasiparticle / hole

$\boxed{\Delta E \Delta t \gtrsim \frac{\hbar}{2}}$

One proves that $\boxed{\frac{1}{\tau_{\vec{k}}} \sim a(|\vec{k}|-k_F)^2 + b T^2 \ll |\xi_{\vec{k}}|}$
due to the Pauli blocking (problem to solve)

Switching on the interaction

$$\boxed{\hat{H}_{int}(t) = \hat{H}_{int}(t=0)\, e^{\eta t}}$$



at $t \to -\infty$ we start from non-interacting gas.

Quasiparticles are observable if their life-time $\tau_{\vec{k}}$ is larger than η^{-1} and $\frac{1}{|\xi_{\vec{k}}|}$ is smaller than η^{-1}

$$\boxed{\frac{1}{\tau_{\vec{k}}} \ll \eta \ll |\xi_{\vec{k}}|}$$

*) Adiabatic continuity does not exclude the possibility of other (collective) excitations which disapear when $\hat{H}_{int} = 0$.

Formally:

chronological operator

interacting Hamiltonian in interaction representation

(*) $$\boxed{\hat{u} = T\, e^{-i\int_{-\infty}^{0} \hat{H}_{int,I}(t)\,dt} \qquad \hat{u}\,\hat{u}^{+} = \hat{1}}$$

time evolution operator

For general state $|\Psi_0\rangle$

$$\hat{a}^{+}_{\vec{k}\sigma}|\Psi_0\rangle \longrightarrow \hat{u}\,\hat{a}^{+}_{\vec{k}\sigma}|\Psi_0\rangle = \underbrace{\left(\hat{u}\,\hat{a}^{+}_{\vec{k}\sigma}\,\hat{u}^{+}\right)}_{\hat{q}^{+}_{\vec{k}\sigma}}\underbrace{\left(\hat{u}|\Psi_0\rangle\right)}_{|\phi\rangle}$$

If evolution is adiabatic (no phase variations)

$$\boxed{\begin{aligned} |\phi\rangle &= \hat{u}\,|FS\rangle \\ \hat{q}^{+}_{\vec{k}\sigma} &= \hat{u}\,\hat{a}^{+}_{\vec{k}\sigma}\,\hat{u}^{+} \end{aligned}}$$

interacting ground state

quasiparticle operator

The Landau Fermi liquid theory is valid when this infinite series is converged

- known proofs in some cases $d=2$ (J. Feldman, M. Knörrer, E. Trubowitz)
- explicit (approximate) construction of $\hat{u}$ would be interesting (open problem (?))

www.math.ubc.ca/~feldman/fl.html

(*) $\hat{u}(t,t_0) = T e^{-i\int_{t_0}^{t} dt' \hat{V}_I(t')} =$

$$= \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{1}{i}\right)^n \int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 \cdots \int_{t_0}^{t} dt_n\, T\left(\hat{V}_I(t_1)\cdots\hat{V}_I(t_n)\right)$$

§3. LANDAU ENERGY FUNCTIONAL

The change in the energy due to a change

$\delta n_{\vec{k}\sigma} = n_{\vec{k}\sigma} - n^0_{\vec{k}\sigma}$ is

$$\delta E[\delta n_{\vec{k}\sigma}] = \sum_{\vec{k}\sigma} \varepsilon_{\vec{k}}\, \delta n_{\vec{k}\sigma}$$

energy of a single-particle added / removed to / from the ground state of the system

Landau suggested to add the next term in this expansion to take into account the interaction between quasiparticles

$$\boxed{\delta E[\delta n_{\vec{k}\sigma}] = \sum_{\vec{k}\sigma} \varepsilon_{\vec{k}}\, \delta n_{\vec{k}\sigma} + \frac{1}{2V} \sum_{\substack{\vec{k}\vec{k}' \\ \sigma\sigma'}} f_{\sigma\sigma'}(\vec{k},\vec{k}')\, \delta n_{\vec{k}\sigma}\, \delta n_{\vec{k}'\sigma'}}$$

<u>Landau function</u>

Def.
$$\frac{1}{V} f_{\sigma\sigma'}(\vec{k},\vec{k}') = \left.\frac{\delta^{(2)} E[\delta n_{\vec{k}\sigma}]}{\delta n_{\vec{k}\sigma}\, \delta n_{\vec{k}'\sigma'}}\right|_{n_{k\sigma} = n^0_{k\sigma}}$$

Symmetric $(\vec{k}\sigma) \leftrightarrow (\vec{k}'\sigma')$

The relevant thermodynamic potential at $T=0$ in grand-canonical ensemble $\Omega(T=0) = E - \mu N$

$$\delta\Omega = \delta E - \mu\,\delta N = \sum_{\vec{k}\sigma} (\varepsilon_{\vec{k}} - \mu)\, \delta n_{\vec{k}\sigma} + \frac{1}{2V} \sum_{\substack{\vec{k}\vec{k}' \\ \sigma\sigma'}} f_{\sigma\sigma'}(\vec{k},\vec{k}')\, \delta n_{\vec{k}\sigma}\, \delta n_{\vec{k}'\sigma'}$$

Since μ is of order δn so both terms are of second-order.

The quasi-particle energy

$$\boxed{\tilde{\varepsilon}_{\bar{k}} = \frac{\delta E[\delta n_{\bar{k}\sigma}]}{\delta n_{\bar{k}\sigma}} = \varepsilon_{\bar{k}} + \frac{1}{V}\sum_{\bar{k}'\sigma'} f_{\sigma\sigma'}(\bar{k},\bar{k}')\,\delta n_{\bar{k}'\sigma'}}$$

$\tilde{\varepsilon}_{\bar{k}} = \tilde{\varepsilon}_{\bar{k}}[\delta n_{\bar{k}\sigma}]$ - a mean-field like, it depends on the other particles in average

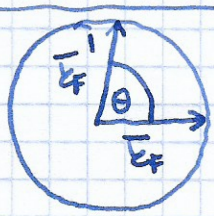
•) Landau parameters

- spin-rotation invariance

$$f_{\sigma\sigma'}(\bar{k},\bar{k}') = f^{s}(\bar{k},\bar{k}') + \sigma\sigma' f^{a}(\bar{k},\bar{k}')$$

- for states near the FS, $|\bar{k}| = |\bar{k}'| = k_F$

isotropic system



$$f_{\sigma\sigma'}(\bar{k},\bar{k}') = f_{\sigma\sigma'}(\bar{k}_F,\bar{k}_F') = f^{s}(\theta) + \sigma\sigma' f^{a}(\theta)$$

$\bar{k}_F = k_F\hat{k}$

$\bar{k}_F' = k_F\hat{k}'$

Expanding in Legendre polynomials

$$\boxed{f^{s,a}(\theta) = \sum_{l=0}^{\infty} f_l^{s,a} P_l(\cos\theta)}$$

$$f_l^{s,a} = (2l+1)\int_0 \frac{d\Omega}{4\pi} f^{s,a}(\theta) P_l(\cos\theta)$$

dimensionless Landau parameters

$$\boxed{F_l^{s,a} = 2\nu^d(0)\, f_l^{s,a}}$$

•) Entropy and thermodynamic potential

Fermi - Dirac statistics of quasiparticles

$$S[n_{\vec{k}}] = -\sum_{\vec{k}\sigma}\left[n_{\vec{k}\sigma}\ln n_{\vec{k}\sigma} + (1-n_{\vec{k}\sigma})\ln(1-n_{\vec{k}\sigma})\right]$$

Thermodynamic potential $\qquad N = \sum_{\vec{k}\sigma} n_{\vec{k}\sigma}$

$$\Omega[n_{\vec{k}\sigma}] = E[n_{\vec{k}\sigma}] - \mu N[n_{\vec{k}\sigma}] - T S[n_{\vec{k}\sigma}]$$

In equilibrium $\quad \dfrac{\delta\Omega}{\delta n_{\vec{k}\sigma}} = 0 \rightarrow \overline{n}_{\vec{k}\sigma}$

$$\overline{n}_{\vec{k}\sigma} = n_F(\tilde{\xi}_k) \qquad \tilde{\xi}_k = \tilde{\varepsilon}_{\vec{k}} - \mu$$

$$\tilde{\varepsilon}_{\vec{k}} = \frac{dE}{dn_{\vec{k}\sigma}}\bigg|_{\overline{n}_{k\sigma}} = \varepsilon_{\vec{k}} + \frac{1}{V}\sum_{\vec{k}'\sigma'} f_{\sigma\sigma'}(\vec{k},\vec{k}')(\overline{n}_{\vec{k}'\sigma'} - n^0_{\vec{k}'\sigma'})$$

Expanding Ω around this equilibrium

$$\Omega[\overline{n}+\delta n] - \Omega[\overline{n}] = \frac{1}{2}\sum_{\substack{\vec{k}\vec{k}'\\ \sigma\sigma'}}\left[-\frac{\delta_{\sigma\sigma'}\delta_{\vec{k}\vec{k}'}}{n_F'(\tilde{\xi}_k)} + \frac{1}{V}f_{\sigma\sigma'}(\vec{k},\vec{k}')\right]\delta n_{\vec{k}\sigma}\delta n_{\vec{k}'\sigma'}$$

- no linear term in $\delta n_{\vec{k}\sigma}$, $\Omega[n]$ - stationary at $n=\overline{n}$
- the first term is from

$$\frac{\delta^{(2)} S[n]}{\delta n_{\vec{k}\sigma}\,\delta n_{\vec{k}'\sigma'}}\bigg|_{\overline{n}} = -\frac{\delta_{\vec{k}\vec{k}'}\delta_{\sigma\sigma'}}{\overline{n}_{\vec{k}\sigma}(1-\overline{n}_{\vec{k}\sigma})} = \frac{\delta_{\vec{k}\vec{k}'}\delta_{\sigma\sigma'}\,\beta}{n_F'(\xi_k')}$$

- Landau function

$$\boxed{\frac{1}{V}f_{\sigma\sigma'}(\vec{k},\vec{k}') = \frac{\delta_{\sigma\sigma'}\delta_{\vec{k}\vec{k}'}}{n_F'(\tilde{\xi}_k)} + \frac{\delta^{(2)}\Omega[\overline{n}]}{\delta n_{\vec{k}\sigma}\,\delta n_{\vec{k}'\sigma'}}\bigg|_{\overline{n}}}$$

allows to derive $f_{\sigma\sigma'}(k,k')$ microscopically.