Functionals of the

•) Entropy and thermodynamic potential

Fermi-Dirac statistics of quasiparticles

$$\boxed{S[n_{\vec{k}}] = -\sum_{\vec{k}\sigma}\left[n_{\vec{k}\sigma}\ln n_{\vec{k}\sigma} + (1-n_{\vec{k}\sigma})\ln(1-n_{\vec{k}\sigma})\right]}$$

Thermodynamic potential $\qquad N = \sum_{\vec{k}\sigma} n_{\vec{k}\sigma}$

$$\boxed{\Omega[n_{\vec{k}\sigma}] = E[n_{\vec{k}\sigma}] - \mu N[n_{\vec{k}\sigma}] - T S[n_{\vec{k}\sigma}]}$$

In equilibrium $\quad \dfrac{\delta\Omega}{\delta n_{\vec{k}\sigma}} = 0 \rightarrow \bar{n}_{\vec{k}\sigma}$

$$\bar{n}_{\vec{k}\sigma} = n_F(\tilde{\xi}_k) \qquad \tilde{\xi}_k = \tilde{\varepsilon}_{\vec{k}} - \mu$$

$$\tilde{\varepsilon}_{\vec{k}} = \left.\frac{dE}{dn_{\vec{k}\sigma}}\right|_{\bar{n}_{k\sigma}} = \varepsilon_{\vec{k}} + \frac{1}{V}\sum_{\vec{k}'\sigma'} f_{\sigma\sigma'}(\vec{k},\vec{k}')(\bar{n}_{\vec{k}'\sigma'} - n^0_{\vec{k}'\sigma'})$$

Expanding Ω around this equilibrium

$$(*)\quad \Omega[\bar{n}+\delta n] - \Omega[\bar{n}] = \frac{1}{2}\sum_{\substack{\vec{k}\vec{k}'\\ \sigma\sigma'}}\left[-\frac{\delta_{\sigma\sigma'}\delta_{\vec{k}\vec{k}'}}{n_F'(\tilde{\xi}_k)} + \frac{1}{V} f_{\sigma\sigma'}(\vec{k},\vec{k}')\right]\delta n_{\vec{k}\sigma}\delta n_{\vec{k}'\sigma'}$$

- no linear term in $\delta n_{\vec{k}\sigma}$, $\Omega[n]$ - stationary at $n=\bar{n}$
- the first term is from

$$\left.\frac{\delta^{(2)} S[n]}{\delta n_{\vec{k}\sigma}\,\delta n_{\vec{k}'\sigma'}}\right|_{\bar{n}} = -\frac{\delta_{\vec{k}\vec{k}'}\delta_{\sigma\sigma'}}{\bar{n}_{k\sigma}(1-\bar{n}_{\vec{k}\sigma})} = \frac{\delta_{\vec{k}\vec{k}'}\delta_{\sigma\sigma'}\beta}{n_F'(\tilde{\xi}_k)}$$

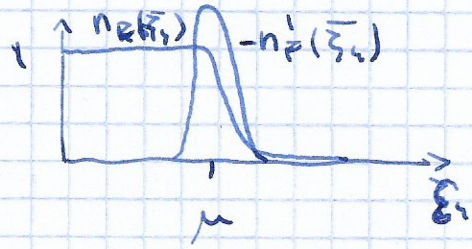
- Landau function

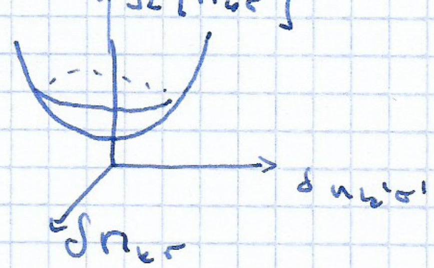
$$\boxed{\frac{1}{V} f_{\sigma\sigma'}(\vec{k},\vec{k}') = \frac{\delta_{\sigma\sigma'}\delta_{\vec{k}\vec{k}'}}{n_F'(\tilde{\xi}_k)} + \left.\frac{\delta^{(2)}\Omega[\bar{n}]}{\delta n_{\vec{k}\sigma}\,\delta n_{\vec{k}'\sigma'}}\right|_{\bar{n}}}$$

allows to derive $f_{\sigma\sigma'}(k,k')$ microscopically.

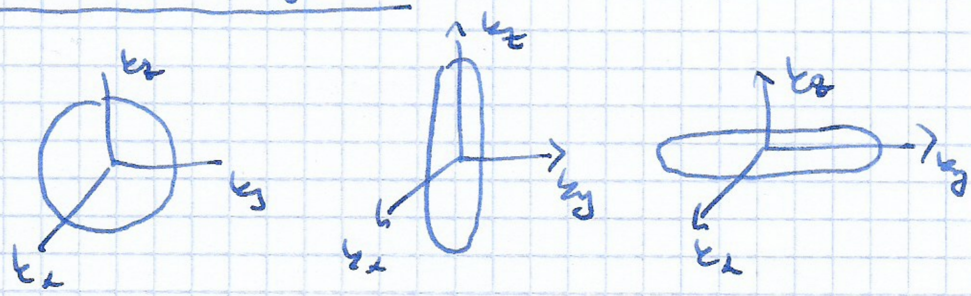
•) Stability of the ground state

Because of the factor $\frac{1}{n_F'(\bar{\xi}_k)}$ in we see that small variations of Ω are due to quasi-particle excitations lying in the thermal broadening of the Fermi surface ($|\bar{\xi}_k| \lesssim T$).

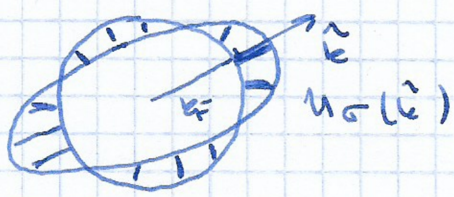




When $T \to 0$ these excitations have vanishing energy and are viewed as displacement of the Fermi surface



Let $\hat{k}$ - direction in which the Fermi momentum k_F varies by infinitesimal amount $u_\sigma(\hat{k})$ for spin-σ particles



Hence,

$$\delta n_{\bar{k}\sigma} = \lim_{u_\sigma(\hat{k}) \to 0} \left\{ n_F[\tilde{\xi}_{\bar{k}} - v_F^* u_\sigma(\hat{k})] - n_F(\bar{\xi}_k) \right\} =$$

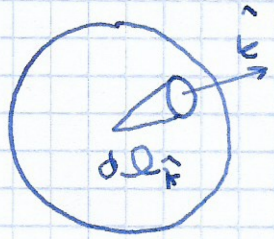
$$= - v_F^* n_F'(\tilde{\xi}_{\bar{k}}) u_\sigma(\hat{k}) \xrightarrow[T \to 0]{} v_F^* \delta(\tilde{\xi}_{\bar{k}}) u_\sigma(\hat{k})$$

$T \to 0$
$\tilde{\xi}_k \to \xi_k$
$n_F'(x) \to -\delta(x)$

The thermodynamical potential change is

$$\delta\Omega[u]_{T\to 0} = V \frac{v_F^{*2} N^*(0)}{2} \sum_{\sigma\sigma'} \left\{ \delta_{\sigma\sigma'} \int \frac{d\Omega_{\hat{k}}}{4\pi} u_\sigma^2(\hat{k}) + \right.$$

$$\left. + \frac{1}{2} \int \frac{d\Omega_{\hat{k}}}{4\pi} \int \frac{d\Omega_{\hat{k}'}}{4\pi} F_{\sigma\sigma'}(\hat{k}_F, \hat{k}_F') u_\sigma(\hat{k}) u_{\sigma'}(\hat{k}') \right\}$$



solid angle in $\vec{k}$-space

Expanding

$$u_\sigma(\hat{k}) = u^s(\hat{k}) + \sigma u^a(\hat{k}) =$$

$$= \sum_{l=0}^{\infty} \sum_{m=-l}^{l} (u^s_{lm} + \sigma u^a_{lm}) Y_l^m(\hat{k})$$

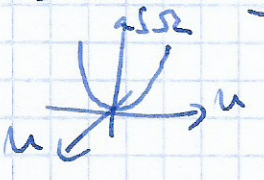
$u_\sigma(\hat{k}) \in \mathbb{R} \to$

$\to u^{s,a}_{l,-m} = (-1)^m u^{s,a}_{lm}$

Using properties of harmonics we get

$$\boxed{\delta\Omega[u] = V \frac{v_F^{*2} N^*(0)}{4\pi} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left[|u^s_{lm}|^2 \left(1 + \frac{F_l^s}{2l+1}\right) + |u^a_{lm}|^2 \left(1 + \frac{F_l^a}{2l+1}\right) \right]}$$

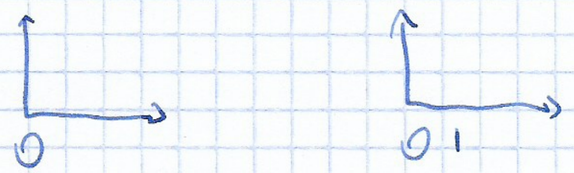
Stability conditions | $\delta\Omega[u]$ form positively defined



$$\boxed{\begin{array}{l} F_l^s > -(2l+1) \\ F_l^a > -(2l+1) \end{array}}$$

If not $\to$ $\boxed{\text{Pomeranchuk instabilities}}$ $\to$ examples

·) <u>Effective mass</u>



O' moves with $\bar v$ with respect to O

$$\bar v = \frac{\bar q}{m} \qquad (\hbar = 1)$$

In a moving frame (Galilean transformation)

$$\hat H' = \hat H - \bar P \cdot \bar v + O(v^2)$$

M total mass, $\bar P$ - total momentum with respect to O

In a translationally invariant system momentum coincides with the current

$$\bar J = \frac{\bar P}{m} = -\frac{1}{m}\frac{\partial E}{\partial \bar v}\Big|_{v=0} = -\frac{\partial E}{\partial \bar q}\Big|_{\bar q=0}$$

E - energy in the moving O' frame

Consider a quasiparticle of momentum $\bar k$ (in Lab. frame O) added to the ground state

$$\vec j_{\bar k} = \frac{\bar k}{m}$$

Compute this current in a moving frame O' where quasiparticle has momentum $\bar k - \bar q$ and the Fermi sea is shifted

$$n_{\bar k} = n^0_{\bar k + \bar q}$$

The quasiparticle energy in O' frame is

$$\tilde{\varepsilon}_{\vec{k}-\vec{p}} = \varepsilon_{\vec{k}-\vec{p}} + \frac{1}{V}\sum_{\vec{k}'\sigma'} f_{\sigma\sigma'}(\vec{k},\vec{k}')\left[n^{0}_{\vec{k}'+\vec{p}} - n^{0}_{\vec{k}'}\right]$$

Hence

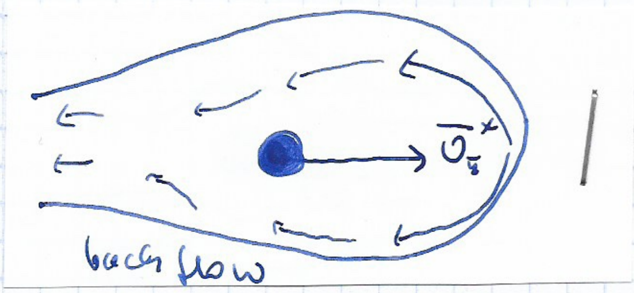
$$\vec{j}_{\vec{k}} = \frac{\vec{k}}{m} = -\frac{\partial \tilde{\varepsilon}_{\vec{k}-\vec{p}}}{\partial \vec{p}}\Big|_{\vec{p}=0} =$$

$$= \vec{\nabla}_{\vec{k}}\,\varepsilon_{\vec{k}} - \frac{1}{V}\sum_{\vec{k}'\sigma'} f_{\sigma\sigma'}(\vec{k},\vec{k}')\,\vec{\nabla}_{\vec{k}'}\, n^{0}_{\vec{k}'} =$$

$$= \vec{v}^{*}_{\vec{k}} + \underbrace{\frac{1}{V}\sum_{\vec{k}'\sigma'} f_{\sigma\sigma'}(\vec{k},\vec{k}')\,\vec{v}^{*}_{\vec{k}'}\,\delta(\xi_{\vec{k}'})}$$

contribution of a localized wave packet of one extra particle moving with $\vec{v}^{*}_{\vec{k}}$

drag current due to the interaction of the quasiparticle with the medium



Near the Fermi surface $\xi_{\vec{k}} \simeq v_F^{*}(|\vec{k}|-k_F)$, $\vec{v}^{*}_{\vec{k}} \simeq v_F \hat{k}$

$$\vec{j}_{\vec{k}} = v_F^{*}\hat{k}\left[1 + 2N(0)\int \frac{d\Omega_{\vec{k}'}}{4\pi} f^{s}(\theta)\cos\theta\right] = v_F^{*}\hat{k}\left(1+\frac{F_1^{s}}{3}\right)$$

$$v_F^{*} = \frac{k}{m^{*}}$$

Hence

$$\boxed{\frac{m^{*}}{m} = 1 + \frac{F_1^{s}}{3}}$$

$$\boxed{\begin{array}{l} \dfrac{m^{*}}{m} > 1 \Leftrightarrow F_1^{s} > 0 \\[2ex] \dfrac{m^{*}}{m} < 1 \Leftrightarrow -3 < F_1^{s} < 0 \end{array}}$$

$$\boxed{F_1^{s} < -3}$$ system unstable

attractive interaction leads to mechanical collapse

§ 4. THERMODYNAMICS

•) specific heat

$$C_V = \left.\frac{\partial E}{\partial T}\right|_{V,N}$$

where
$$E = E(T=0) + \sum_{k\sigma} \varepsilon_k \,\delta n_{k\sigma} + \frac{1}{2V}\sum_{\substack{kk'\\ \sigma\sigma'}} f_{\sigma\sigma'}(\vec{k},\vec{k}')\,\delta n_{k\sigma}\,\delta n_{k'\sigma'}$$

and $\delta n_{\vec{k}\sigma} = n_F(\tilde{\varepsilon}_{\vec{k}}) - n^0_{\vec{k}}$

If we neglect the interaction between quasiparticles
$$C_V = V\,\frac{2\pi^2}{3}\,N^*(0) = V\,\frac{m^* k_F}{3}\,T$$

$m^* = (2.8)\, m_{He}$

$m^*_{{}^3He} \simeq 1000\ m_e$

$$\boxed{c_V = \frac{C_V}{V} = \frac{m^* k_F}{3}\,T}$$

the same as for non-interacting fermions but $m \to m^*$

→ Since $\int dk\, k^2\, \delta n_{\vec{k}\sigma} \sim O(T^4)$ the interacting term is $O(T^4)$ and can be neglected in a grand canonical ensemble. Even if we include $\delta\mu \sim O(T^2)$.

→ In a canonical ensemble $f_{\sigma\sigma'}(\vec{k},\vec{k}') \simeq f_{\sigma\sigma'}(\vec{k}_F,\vec{k}_F')$
$$\frac{1}{V}\sum_{k} \delta n_{k\sigma} = \frac{1}{2\pi^2}\int dk\, k^2\, \delta n_{\vec{k}\sigma} = 0 \quad \text{since } N = \text{const.}$$

→ Sommerfeld expansion

$$C_V = \left(\frac{\partial E}{\partial T}\right) = \sum_{k\sigma} \varepsilon_k \left(\frac{\partial n_{\vec{k}\sigma}}{\partial T}\right) \simeq N^*(0)\int_{-\infty}^{\infty} d\varepsilon\, \varepsilon \left(\frac{\partial n_F}{\partial T}\right) =$$

$$= N^*(0)\, T \underbrace{\int_{-\infty}^{\infty} dx\, \frac{x^2}{(e^x+1)(e^{-x}+1)}}_{\pi^2/3} = \frac{\pi^2}{3}\, N^*(0)\, T$$

$$N^*(0) = \frac{m^* k_F}{\pi^2} \qquad (\hbar = k_B = 1)$$

•) Compressibility

Df. $$\boxed{\varkappa = -\frac{1}{V}\frac{\partial V}{\partial p}\Big|_{T,N}}$$ isothermal compressibility

Note, the free energy $F(T,V,N) = E - TS = V f(T,n)$

Textensive $\qquad n = \frac{N}{V}$

$$dF = -S\,dT - p\,dV + \mu\,dN$$

$$S = -\left(\frac{\partial F}{\partial T}\right)_{V,N}, \quad p = -\left(\frac{\partial F}{\partial V}\right)_{T,N}, \quad \mu = \left(\frac{\partial F}{\partial N}\right)_{T,V}$$

$$\Rightarrow \quad \frac{\partial p}{\partial N} = -\frac{\partial^2 F}{\partial N\,\partial V} = -\frac{\partial \mu}{\partial V}$$

$$\frac{\partial p}{\partial n}\frac{\partial n}{\partial N} = \frac{1}{V}\frac{\partial p}{\partial n} \quad \rightarrow \quad \frac{\partial p}{\partial n} = V\frac{\partial p}{\partial N} = -V\frac{\partial \mu}{\partial V}$$

$$\rightarrow \quad \frac{1}{\varkappa} = -V\frac{\partial p}{\partial V} = -V\frac{\partial p}{\partial n}\frac{\partial n}{\partial V} = -V\frac{\partial p}{\partial n}\left(-\frac{N}{V^2}\right) = n\frac{\partial p}{\partial n} =$$

$$= -nV\frac{\partial \mu}{\partial V} = -nV\frac{\partial \mu}{\partial n}\frac{\partial n}{\partial V} =$$

$$= -nV\frac{\partial \mu}{\partial n}\left(-\frac{N}{V^2}\right) = n^2\frac{\partial \mu}{\partial n}$$

$$\Rightarrow \quad \boxed{\varkappa = \frac{1}{n^2}\frac{\partial n}{\partial \mu}\Big|_T}$$

variation of $n = \frac{k_F^3}{3\pi^2}$ is equivalent to a variation of k_F

$$\frac{\partial k_F}{\partial n} = \frac{\pi^2}{k_F^2}$$

When k_F is changed the $\mu = \varepsilon_{k_F}$ is changed too

$$\frac{\partial \mu}{\partial n} = \frac{\partial \varepsilon_{k_F}}{\partial k_F}\frac{\partial k_F}{\partial n} + \frac{1}{V}\sum_{k'\sigma'} f_{\sigma\sigma'}(\bar{k}_F, \bar{k}')\frac{\partial n_{k'\sigma'}}{\partial k_F}\frac{\partial k_F}{\partial n}$$

using $\quad \dfrac{\partial \varepsilon_{k_F}}{\partial k_F} = v_F^* = \dfrac{k_F}{m^*}$

$$\frac{\partial n_{k\sigma}}{\partial k_F} \underset{T=0}{=} \delta(k_F - k)$$

$$\frac{\partial \mu}{\partial n} = \frac{\pi^2}{k_F^2}\left[v_F^* + \sum_{\sigma}\int \frac{d_3 k'}{(2\pi)^3} f_{\sigma\sigma'}(\bar{k}_F, \bar{k}')\,\delta(k_F - k')\right] =$$

$$= \frac{\pi^2}{k_F^2}\left[v_F^* + 2N^*(0)\, v_F^* f_0^s \right] = \frac{1 + F_0^s}{2N^*(0)}$$

Finally,
$$\boxed{\varkappa = \frac{1}{n^2}\frac{2N^*(0)}{1 + F_0^s}}$$

naive result $m \to m^* \quad \to \varkappa = \frac{1}{n^2} 2N^*(0)$ <u>wrong</u>

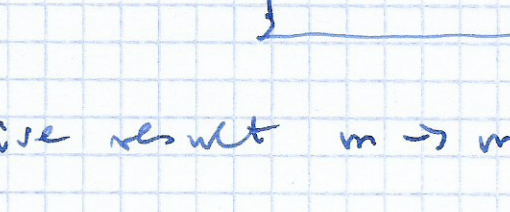
There is a renormalization due to F_0^s.

The <u>stability condition</u> $\boxed{F_0^s > -1}$

•) <u>Spin susceptibility</u>

$\bar{B} = (0, 0, B)$ $\qquad \varepsilon_{\bar{k}\sigma}$ is shifted by $\frac{\sigma}{2} g \mu_B B$

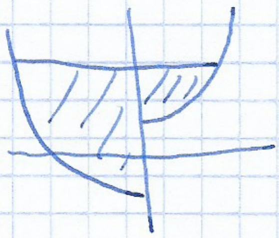
Zeeman shift



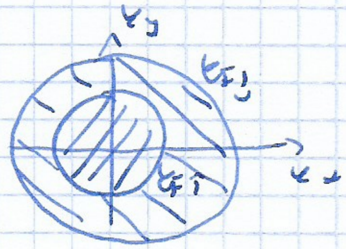
Quasiparticle energy

$$\tilde{\varepsilon}_{k\sigma} = \varepsilon_k + \frac{\sigma}{2} g \mu_B B + \frac{1}{V} \sum_{\vec{k}'\sigma'} f_{\sigma\sigma'}(\vec{k},\vec{k}')\, \delta n_{\vec{k}'\sigma'}$$

Two Fermi surfaces :



$$\mu = \tilde{\varepsilon}_{k_{F\uparrow}} = \tilde{\varepsilon}_{k_{F\downarrow}}$$



$$\varepsilon_{k_{F\uparrow}} + \frac{g\mu_B B}{2} + \frac{1}{V} \sum_{\vec{k}'\sigma'} f_{\uparrow\sigma'}(\vec{k}_F,\vec{k}')\, \delta n_{\vec{k}'\sigma'} =$$

$$= \varepsilon_{k_{F\downarrow}} - \frac{g\mu_B B}{2} + \frac{1}{V} \sum_{\vec{k}'\sigma'} f_{\downarrow\sigma'}(\vec{k}_F,\vec{k}')\, \delta n_{\vec{k}'\sigma'}$$

$$\delta n_{\vec{k}\sigma} = \Theta(k_F + \delta k_{F\sigma} - k) - \Theta(k_F - k) = \delta k_{F\sigma}\, v_F^*\, \delta(\xi_{\vec{k}})$$

$$\delta k_{F\sigma} = k_{F\sigma} - k_F + \mathcal{O}(B^2)$$

using $\varepsilon_{k_{F\sigma}} = \varepsilon_{k_F} + v_F^*\, \delta k_{F\sigma}$

$$\rightarrow \quad v_F^* (\delta k_{F\uparrow} - \delta k_{F\downarrow}) = - \frac{g\mu_B B}{1 + F_0^a}$$

<u>magnetization</u>

$$M = -\frac{1}{2V} g\mu_B \sum_{k} (\delta n_{\vec{k}\uparrow} - \delta n_{\vec{k}\downarrow}) = -\frac{1}{2} g\mu_B N^*(0) v_F^* (\delta k_{F\uparrow} - \delta k_{F\downarrow})$$

$$= \left(\frac{g\mu_B}{2}\right)^2 \frac{2N^*(0)}{1 + F_0^a} B$$

<u>Pauli susceptibility</u>

$$\boxed{\chi = \frac{\partial M}{\partial B} = \left(\frac{g\mu_B}{2}\right)^2 \frac{2N^*(0)}{1 + F_0^a}}$$

$m \to m^*$

plus renormalization

$$\frac{1}{1+F_0^a}$$

Stability $\boxed{F_0^a > -1}$