§7. QUASIPARTICLES FROM GREEN FUNCTIONS

To describe many-body systems it is more convinient to use Green's functions

The single-particle Green's function $T=0$

$$\boxed{iG(\lambda, t; \lambda' t') = \frac{\langle \Psi_0 | T\left(c_\lambda(t)_H \, c^\dagger_{\lambda'}(t')_H \right) | \Psi_0 \rangle}{\underbrace{\langle \Psi_0 | \Psi_0 \rangle}_{\text{normalization}}}}$$

quantum numbers
$\lambda = \bar{k}, \sigma$
$\lambda = \bar{r}, \sigma$
etc.

N-body ground state
$\hat{H} | \Psi_0 \rangle = E_0 | \Psi_0 \rangle$

in Heisenberg picture
$c_\lambda(t)_H = e^{i\hat{H}t} c_\lambda e^{-i\hat{H}t}$ $\boxed{\hbar = 1}$

$$T\left(c_\lambda(t)_H \, c^\dagger_{\lambda'}(t') \right) = \begin{cases} c_\lambda(t) \, c^\dagger_{\lambda'}(t') & t > t' \\ -c^\dagger_{\lambda'}(t') \, c_\lambda(t) & t < t' \end{cases}$$

time order operator
(chronological operator)

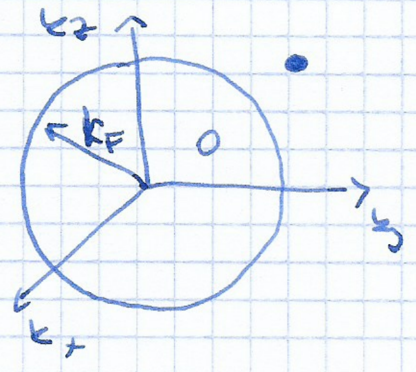
→ For translationally invariant systems $\lambda = \bar{k}$ and $G(\bar{k}, t; \bar{k}' t') = \delta_{kk'} \, G(\bar{k}, t-t')$ and we consider "spinless" fermions only

$$\boxed{iG(\bar{k}, t) = ?}$$

For a non-interacting fermions

$$iG^0(\bar{k},t) = \theta(t)\left[e^{-i\varepsilon_{\bar{k}}t}\,\theta(|\bar{k}|-k_F)\right] \quad \text{particle}$$

$$-\,\theta(-t)\left[e^{-i\varepsilon_{\bar{k}}t}\,\theta(k_F-|\bar{k}|)\right] \quad \text{hole}$$



stationary solution

particle (hole) with $\bar{k}$ is an exact eigenstate of free fermions (infinitely long lived) with a given energy $\varepsilon_{\bar{k}}$

Under which conditions we can have a particle like objects (quasi particles) with

$$G(\bar{k},t) \sim \underbrace{e^{-i\varepsilon(\bar{k})t}}_{\substack{\text{stationary like}\\ \text{with defined}\\ \text{energy } \varepsilon(\bar{k})}} \; \underbrace{e^{-\Gamma(\bar{k})t}}_{\substack{\text{finite life}\\ \text{time (not}\\ \text{exact eigenstate)}\\ \text{but long}\\ \text{enough (!)}}}$$

Firstly, we look into the general structure of the single-particle Green's function.

With the help of a unity resolution in Fock space

$$1 = |vac\rangle\langle vac| + \sum_n |\psi_n^1\rangle\langle\psi_n^1| + \sum_n |\psi_n^2\rangle\langle\psi_n^2| + \dots \sum_n |\psi_n^N\rangle\langle\psi_n^N| + \dots$$

and Fourier transforming

$$iG(\vec{k},t) = i\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} G(\vec{k},\omega)$$

we find Lehmann-representation of the Green function

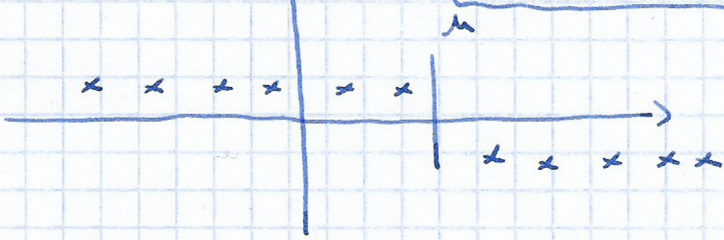
$$\boxed{G(\vec{k},\omega) = \lim_{\eta\to 0+}\left[\sum_n \frac{|\langle\psi_n^{N+1}(\vec{k})|c_{\vec{k}}^\dagger|\psi_0^N\rangle|^2/\langle\psi_0^N|\psi_0^N\rangle}{\omega-\mu-W_{n,\vec{k}}^{N+1}+i\eta} + \sum_n \frac{|\langle\psi_n^{N-1}(-\vec{k})|c_{\vec{k}}|\psi_0^N\rangle|^2/\langle\psi_0^N|\psi_0^N\rangle}{\omega-\mu+W_{n,-\vec{k}}^{N-1}+i\eta}\right]}$$

Here

$$\boxed{W_{n,\pm\vec{k}}^{N\pm 1} \equiv E_n^{N\pm 1}(\pm\vec{k}) - E_0^{N\pm 1} > 0}$$

There are poles at

$$\boxed{\begin{aligned} \omega &= \mu + W_{n,\vec{k}}^{N+1} - i\eta \\ \omega &= \mu - W_{n,-\vec{k}}^{N-1} + i\eta \end{aligned}}$$



We also introduce <u>retarded</u> and <u>advanced</u> Green's functions

$$i\,G^R(\vec{r}t,\vec{r}'t') \equiv \theta(t-t')\,\frac{\langle\Psi_0|\{\hat{\psi}(\vec{r}t)_H,\hat{\psi}^\dagger(\vec{r}'t')_H\}|\Psi_0\rangle}{\langle\Psi_0|\Psi_0\rangle}$$

$$i\,G^A(\vec{r}t,\vec{r}'t') \equiv -\theta(t'-t)\,\frac{\langle\Psi_0|\{\hat{\psi}(\vec{r}t)_H,\hat{\psi}^\dagger(\vec{r}'t')_H\}|\Psi_0\rangle}{\langle\Psi_0|\Psi_0\rangle}$$

For which we find a corresponding Lehmann rep.

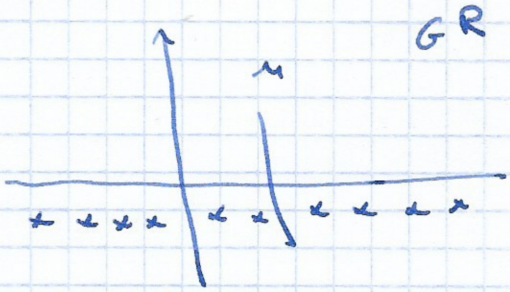
$$G^{R(A)}(\vec{k},\omega) = \lim_{\eta\to0^+}\Bigg[\sum_n \frac{|\langle\Psi_n^{N+1}(\vec{k})|c^\dagger_{\vec{k}}|\Psi_0^N\rangle|^2/\langle\Psi_0^N|\Psi_0^N\rangle}{\omega-\mu-W^{N+1}_{n,\vec{k}}\pm i\eta} +$$

$$+\sum_n \frac{|\langle\Psi_n^{N-1}(-\vec{k})|c_{\vec{k}}|\Psi_0^N\rangle|^2/\langle\Psi_0|\Psi_0\rangle}{\omega-\mu+W^{N-1}_{n,-\vec{k}}\pm i\eta}\Bigg]$$

with poles at (R)

$$\omega = \mu + W^{N+1}_{n,\vec{k}} - i\eta$$
$$\omega = \mu - W^{N-1}_{n,-\vec{k}} - i\eta$$

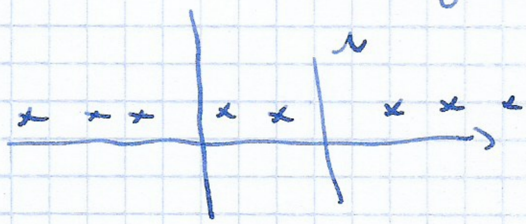
G^R



G^A

$$\omega = \mu + W^{N+1}_{n,\vec{k}} + i\eta$$

$$\omega = \mu - W^{N-1}_{n,-\vec{k}} + i\eta$$



G^R – analytic in upper ω-plane

G^A – analytic in lower ω-plane

<u>Note</u>

$$G^R(\vec{k},\omega)^* = G^A(\vec{k},\omega) \quad \text{for } \omega\in\mathbb{R}$$

for $\mu\neq\omega$ and $\mu\in\mathbb{R}$, $\eta\to0^+$

$\mu<\omega$: $G(\vec{k},\omega) = G^R(\vec{k},\omega)$

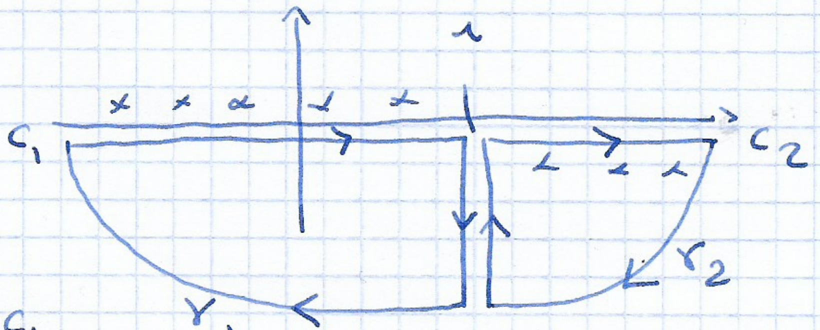
$\mu>\omega$: $G(\vec{k},\omega) = G^A(\vec{k},\omega)$

Let's do the integral to check a time evolution

$$G(\bar{k},t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} G(\bar{k},t) =$$

$$= \int_{-\infty}^{\mu} \frac{d\omega}{2\pi} e^{-i\omega t} G^{A}(\bar{k},\omega) + \int_{\mu}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} G^{R}(\bar{k},\omega)$$

and evaluate it on the complex ω-space



This choice of contour means that we are dealing with quasiparticles $t>0$. For qh, $t<0$

no poles in C_1

$$0 = \int_{C_1} \frac{d\omega}{2\pi} e^{-i\omega t} G^{A}(\bar{k},\omega) = \int_{-\infty}^{\mu} \frac{d\omega}{2\pi} e^{-i\omega t} G^{A}(\bar{k},\omega) +$$

$$+ \int_{\mu}^{\mu-i\infty} \frac{d\omega}{2\pi} e^{-i\omega t} G^{A}(\bar{k},\omega) + \underbrace{\int_{\gamma_1} \frac{d\omega}{2\pi} e^{-i\omega t} G^{A}(\bar{k},\omega)}_{\xrightarrow{R\to\infty} 0 \text{ due to } e^{-i\omega t}}$$

Hence,

$$\int_{-\infty}^{\mu} \frac{d\omega}{2\pi} e^{-i\omega t} G^{A}(\bar{k},\omega) = \int_{\mu-i\infty}^{\mu} \frac{d\omega}{2\pi} e^{-i\omega t} G^{A}(\bar{k},\omega)$$

The second term

$$\int_{C_2} \frac{d\omega}{2\pi} e^{-i\omega t} G^R(\vec{k},\omega) = \int_{\mu}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} G^R(\vec{k},\omega) +$$

$$+ \underbrace{\int_{\gamma_2} \frac{d\omega}{2\pi} e^{-i\omega t} G^R(\vec{k},\omega)}_{\xrightarrow{R\to\infty} 0} + \int_{\mu - i\infty}^{\mu} \frac{d\omega}{2\pi} e^{-i\omega t} G^R(\vec{k},\omega) =$$

$$= \underbrace{-i \sum_{\nu} z_\nu e^{-i\omega_\nu t}}$$

contributions from poles below $\mathbb{R}_+$

Thus,

$$\int_{\mu}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} G^R(\vec{k},\omega) = -i \sum_{\nu} z_\nu e^{-i\omega_\nu t} - \int_{\mu-i\infty}^{\mu} \frac{d\omega}{2\pi} e^{-i\omega t} G^R(\vec{k},\omega)$$

Separating $\boxed{\omega_\nu = \varepsilon_\nu - i\Gamma_\nu}$

we obtain exact result

$$\boxed{G(\vec{k},t) = \underbrace{-i \sum_{\nu} z_\nu e^{-i\varepsilon_\nu t} e^{-\Gamma_\nu t}} + \underbrace{\int_{\mu-i\infty}^{\mu} \frac{d\omega}{2\pi} e^{-i\omega t} \left[G^A(\vec{k},\omega) - G^R(\vec{k},\omega)\right]}}$$

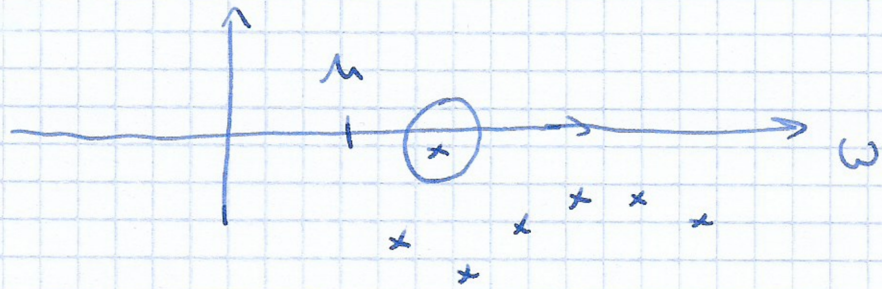
Stationary like oscillations with finite time excitations with $\omega_\nu > \mu$ quasiparticles $(t > 0)$

Some background contribution (incoherent)

Consider the pole

$$\omega_0 = \varepsilon_0 - i\Gamma_0$$

which is closest to $\mathbb{R}_+$, i.e. $\underline{\Gamma_0 < \Gamma_\nu}$, $\nu \neq 0$

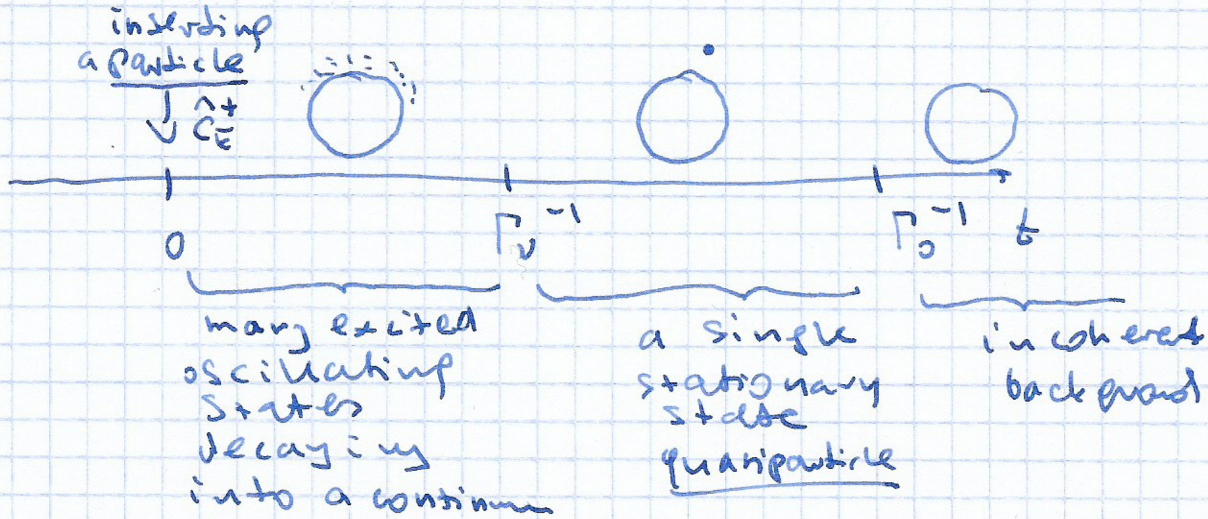


For sufficiently large times we can write

$$-i\sum_\nu Z_\nu e^{-i\varepsilon_\nu t} e^{-\Gamma_\nu t} \simeq -i Z_0 e^{-i\varepsilon_0 t} e^{-\Gamma_0 t}$$

This time t cannot be too long, i.e.

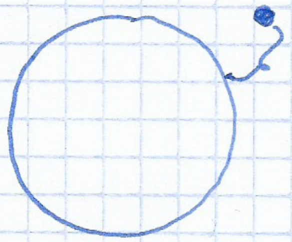
$$\frac{1}{\Gamma_\nu} < t \lesssim \frac{1}{\Gamma_0}$$



Therefore

$$\boxed{G(\vec{k},t) \simeq -i Z_0 e^{-i\varepsilon_0 t} e^{-\Gamma_0 t} + \int_{\mu-i\infty}^{\mu} \frac{d\omega}{2\pi} e^{-i\omega t}\left[G^A(\vec{k},\omega) - G^R(\vec{k},\omega)\right]}$$

Under which condition the background integral can be neglected?



$(\varepsilon_0 - \mu) > 0$

measure of energy with respect to ε_F

In order to guarantee a resonably well-defined energy, the quasiparticle must propagate at least for a time $t \gg \frac{1}{\varepsilon_0 - \mu}$.

$$\boxed{\begin{array}{c} \Delta\tau \Delta\varepsilon \gtrsim 1 \\ t \gg \Delta\tau \end{array}}$$

The conditions

$$\boxed{\begin{array}{c} \frac{1}{\Gamma_\nu} < t \lesssim \frac{1}{\Gamma_0} \\ \frac{1}{\varepsilon_0 - \mu} \ll t \lesssim \frac{1}{\Gamma_0} \end{array}}$$

are sufficient conditions to neglect the background

From the second $\rightarrow$ $\boxed{\Gamma_0 \ll \varepsilon_0 - \mu}$

the life time $\frac{1}{\Gamma_0}$ of the quasi-particle must be large enough for an accurate its measurement. The pole must be very close to $\mathbb{R}_+$.

We estimate the background part now

It vanishes for large Im ω. So we estimate it for $\omega = \omega_{real} \in \mathbb{R}$

$$G^{R/A}(\bar{\epsilon}, \omega) \simeq G^{R/A}(\bar{\epsilon}, \omega_{real})$$

Then the pole ω_0 dominates the integral

$$G^R(\bar{\epsilon}, \omega) \simeq \frac{z_0}{\omega - \epsilon_0 + i\Gamma_0} = G^A(\bar{\epsilon}, \omega)^*$$

Now,

$$I \equiv \int_{\mu - i\infty}^{\mu} \frac{d\omega}{2\pi} e^{-i\omega t} \left[G^A(\bar{\epsilon}, \omega) - G^R(\bar{\epsilon}, \omega) \right] =$$

$$\simeq 2 i z_0 \Gamma_0 \int_{\mu - i\infty}^{\mu} \frac{d\omega}{2\pi} \frac{e^{-i\omega t}}{(\omega - \epsilon_0)^2 + \Gamma_0^2} = \left\{ \begin{array}{l} y = i(\omega - \mu) \\ \omega = \mu - iy \\ d\omega = -i\,dy \end{array} \right\} =$$

$$= -\frac{z_0 \Gamma_0}{\pi} e^{-i\mu t} \int_0^{\infty} dy \frac{e^{-yt}}{\Gamma_0^2 + (\epsilon_0 - \mu + iy)^2}$$

If $t \gg \frac{1}{\epsilon_0 - \mu}$ then e^{-yt} is different from zero essentially only if $y \ll \epsilon_0 - \mu$. But in this regime

$$\Gamma_0^2 + (\epsilon_0 - \mu + iy)^2 \simeq \Gamma_0^2 + (\epsilon_0 - \mu)^2 \simeq (\epsilon_0 - \mu)^2$$

So

$$I \simeq -\frac{z_0 \Gamma_0}{\pi} e^{-i\mu t} \int_0^{\infty} dy \frac{e^{-yt}}{(\epsilon_0 - \mu)^2} = -\frac{z_0 \Gamma_0}{\pi} \frac{e^{-i\mu t}}{t (\epsilon_0 - \mu)^2} \simeq$$

$$\simeq -\frac{z_0}{\pi} e^{-i\mu t} \underbrace{t^{-2} (\epsilon_0 - \mu)^{-2}}_{\ll 1 \text{ for } t \simeq \frac{1}{\Gamma_0}}$$

So $|I| \ll 1$

whereas qp part $\left| -i z_0 e^{-i\epsilon_0 t} e^{-\Gamma_0 t} \right|_{t \simeq \frac{1}{\Gamma_0}} \simeq |z_0| \gg |I|$

Thus for $t>0$ we obtained a quasiparticle picture

$$G(\bar{k},t) \simeq -i\, Z(\bar{k})\, e^{-i\varepsilon(\bar{k})t}\, e^{-\Gamma(\bar{k})t} \qquad t>0$$

Similarly we can get quasihole picture.

Summary

$$\boxed{\begin{aligned} G(\bar{k},t) = -i\, Z(\bar{k}) \Big[& \theta(t)\,\theta(\varepsilon(\bar{k})-\mu)\, e^{-i\varepsilon(\bar{k})t}\, e^{-\Gamma(\bar{k})t} + \\ & + \theta(-t)\,\theta(\mu-\varepsilon(\bar{k}))\, e^{-i\varepsilon(\bar{k})t}\, e^{-\Gamma(\bar{k})t} \Big] + I(\bar{k},t) \end{aligned}}$$

quasi stationary states with finite life-time

Spectral function view

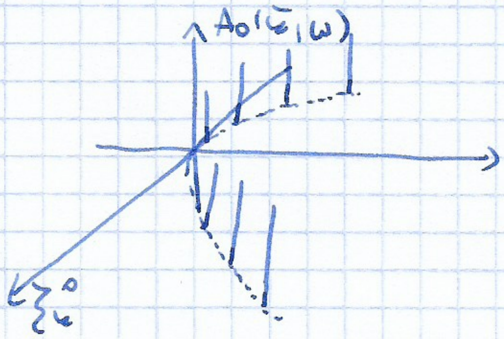
→ **Fermi gas:** $G^R_0(\vec{k},\omega) = \dfrac{1}{\omega - \underbrace{\varepsilon^0_{\vec{k}} + \mu}_{\xi^0_{\vec{k}} \equiv \varepsilon^0_{\vec{k}} - \mu} + i\eta}$

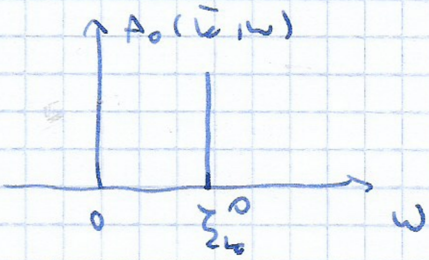
$$G^R_0(\vec{k},t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{e^{-i\omega t}}{\omega + i\eta - \xi^0_{\vec{k}}} =$$

$$= -i\,\theta(t)\, e^{-i\xi^0_{\vec{k}} t}$$

Spectral function

$$A_0(\vec{k},\omega) = -\frac{1}{\pi}\,\mathrm{Im}\, G^R_0(\vec{k},\omega) = \delta(\omega - \xi^0_{\vec{k}})$$





→ **Fermi liquid:**

$$\boxed{G_R(\vec{k},\omega) = \frac{1}{\omega + i\eta - \xi^0_{\vec{k}} - \Sigma^R(\vec{k},\omega)}}$$

Self-energy

Spectral function

$$\boxed{A(\vec{k},\omega) = -\frac{1}{\pi}\,\frac{\mathrm{Im}\,\Sigma^R(\vec{k},\omega)}{\left(\omega - \xi^0_{\vec{k}} - \mathrm{Re}\,\Sigma^R(\vec{k},\omega)\right)^2 + \left(\mathrm{Im}\,\Sigma^R(\vec{k},\omega)\right)^2}}$$

When $\mathrm{Im}\,\Sigma^R(\vec{k},\omega) \to 0$ it reduces to the δ-function

If $\mathrm{Im}\,\Sigma^R(\vec{k},\omega)$ varies weakly for $\omega \simeq \xi_{\vec{k}}$ the maximum (peak) of spectral function is determined by

$$\boxed{\xi_{\vec{k}} - \xi^0_{\vec{k}} - \mathrm{Re}\,\Sigma^R(\vec{k},\xi_{\vec{k}}) = 0}$$

using $\left.\dfrac{\partial}{\partial\omega}\,\mathrm{Im}\,\Sigma^R(\vec{k},\omega)\right|_{\omega=\xi_{\vec{k}}} = 0$

energy of qp with momentum $\vec{k}$

E.g. at the Fermi momentum k_F, $\xi_{k_F} = 0$

$$\rightarrow \quad \xi^0_{k_F} + \Sigma^R(k_F, 0) = 0$$

$\boxed{\text{since } \Sigma^R(k_F,0) \in \mathbb{R} \text{ at } T=0}$

in interacting system $\mu \neq \epsilon_F^0 = \frac{k_F^2}{2m}$ so $\xi^0_{k_F} \neq 0$.

But $k_F = (3\pi^2 n)^{1/3}$ - Luttinger theorem.

For ω very close to $\xi_{\bar{k}}$:

$$\omega - \xi^0_{\bar{k}} - \mathrm{Re}\, \Sigma^R(\bar{k}, \omega) \simeq \omega - \xi^0_{\bar{k}} - \mathrm{Re}\, \Sigma^R(\bar{k}, \xi_{\bar{k}}) -$$

$$- (\omega - \xi_{\bar{k}}) \frac{\partial}{\partial \omega} \mathrm{Re}\, \Sigma^R(\bar{k}, \omega)\Big|_{\omega = \xi_{\bar{k}}} =$$

$$= \frac{\omega - \xi_{\bar{k}}}{Z_{\bar{k}}}$$

with

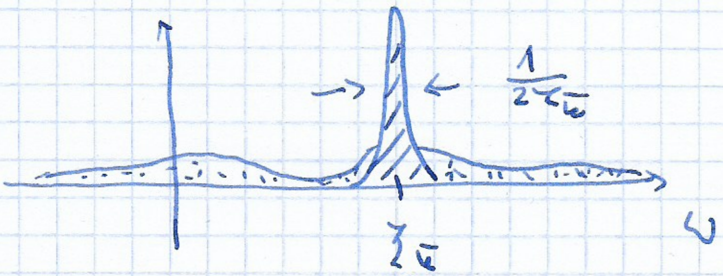
$$\boxed{Z_{\bar{k}} = \frac{1}{1 - \frac{\partial}{\partial \omega} \mathrm{Re}\, \Sigma^R(\bar{k}, \omega)\Big|_{\omega = \xi_{\bar{k}}}}}$$

quasiparticle weight $0 < Z_{\bar{k}} \leq 1$,
Fermi liquid renormalization function.

Hence,

$$\boxed{A(\bar{k}, \omega) = \frac{Z_{\bar{k}}}{\pi} \frac{\frac{1}{2\tau_{\bar{k}}}}{(\omega - \xi_{\bar{k}})^2 + \left(\frac{1}{2\tau_{\bar{k}}}\right)^2} + A_{inc}(\bar{k}, \omega)}$$

$\boxed{\frac{1}{\tau_{\bar{k}}} = -2 Z_{\bar{k}} \,\mathrm{Im}\, \Sigma^R(\bar{k}, \xi_{\bar{k}})}$ inverse of quasiparticle life time



Also, we find

$$\boxed{\begin{aligned} G^R(\vec{k},\omega) &= \int_{-\infty}^{\infty} d\omega' \frac{A(\vec{k},\omega')}{\omega+i\eta-\omega'} = \frac{Z_{\vec{k}}}{\omega-\xi_{\vec{k}}+\frac{i}{2\tau_{\vec{k}}}} + G^R_{inc}(\vec{k},\omega) \\ G^R(\vec{k},\omega) &= -i Z_{\vec{k}}\, \theta(t)\, e^{-i\xi_{\vec{k}}t-\frac{t}{2\tau_{\vec{k}}}} + G^R_{inc}(\vec{k},t) \end{aligned}}$$

$$\frac{1}{\xi_{\vec{k}}} \ll \tau \ll \tau_{\vec{k}}$$

<u>Momentum distribution</u>

$$n_{\vec{k}} = \langle c^+(\vec{k})\, c(\vec{k}) \rangle = G(\vec{k}, \tau=0^-) = \frac{1}{\beta}\sum_{\omega_n} e^{i\omega_n\eta}\, G(\vec{k}, i\omega_n) =$$

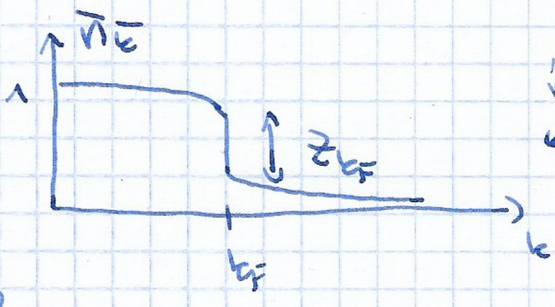
$$= \frac{1}{\beta}\sum_{\omega_n} e^{i\omega_n\eta} \int_{-\infty}^{\infty} d\omega \frac{A(\vec{k},\omega)}{i\omega_n-\omega} =$$

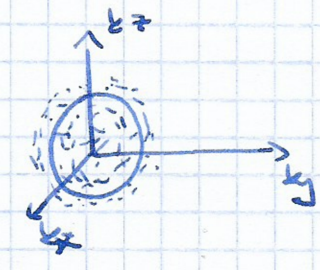
$$= \int_{-\infty}^{\infty} d\omega\, n_F(\omega)\, A(\vec{k},\omega) \underset{T=0}{=} \int_{-\infty}^{0} d\omega\, A(\vec{k},\omega)$$

when $\frac{1}{\tau_{\vec{k}}} \ll \xi_{\vec{k}} \rightarrow A(\vec{k},\omega) \simeq Z_{\vec{k}_F}\, \delta(\omega-\xi_{\vec{k}})$ as $\xi_{\vec{k}}\to 0$

$$\lim_{|\vec{k}|\to k_F^+} n_{\vec{k}} - \lim_{|\vec{k}|\to k_F^-} n_{\vec{k}} = \left[\lim_{|\vec{k}|\to k_F^+} - \lim_{|\vec{k}|\to k_F^-}\right] \int_{-\infty}^{0} d\omega\, Z_{\vec{k}}\, \delta(\omega-\xi_{\vec{k}}) =$$

$$= -Z_{\vec{k}_F}$$





A sharp step at $T=0$ even in interacting fermions if $Z_{k_F} \neq 0$

<u>Fermi liquid condition</u> $\rightarrow$ $\boxed{\text{existence of the Fermi surface}}$