Thus for $t>0$ we obtained a quasiparticle picture

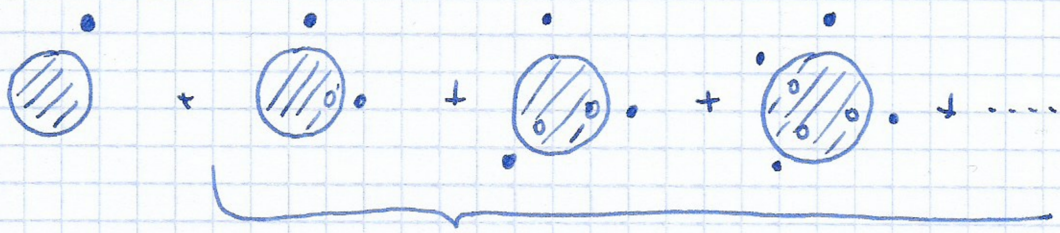
$$G(\vec{k},t) \simeq -i\,Z(\vec{k})\, e^{-i\varepsilon(\vec{k})t}\, e^{-\Gamma(\vec{k})t} \qquad t>0$$

Similarly we can get quasihole picture.

Summary

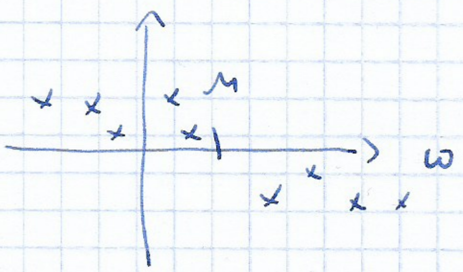
$$\boxed{G(\vec{k},t) = -i\,Z(\vec{k})\Big[\theta(t)\,\theta(\varepsilon(\vec{k})-\mu)\, e^{-i\varepsilon(\vec{k})t}\, e^{-\Gamma(\vec{k})t} + \theta(-t)\,\theta(\mu-\varepsilon(\vec{k}))\, e^{-i\varepsilon(\vec{k})t}\, e^{-\Gamma(\vec{k})t}\Big] + I(\vec{k},t)}$$

quasi stationary states with finite life-time



$|\Psi_{\text{exact}}\rangle = |\Psi_{QP}\rangle + |\Psi_{inc}\rangle$

$$\frac{1}{\Gamma_?} < t \lesssim \frac{1}{\Gamma_0}$$



$$\frac{1}{\varepsilon_0-\mu} \ll t \lesssim \frac{1}{\Gamma_0}$$

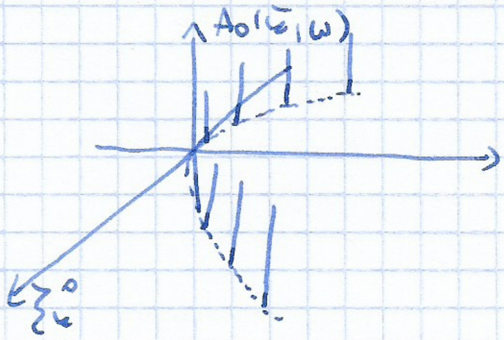
Spectral function view

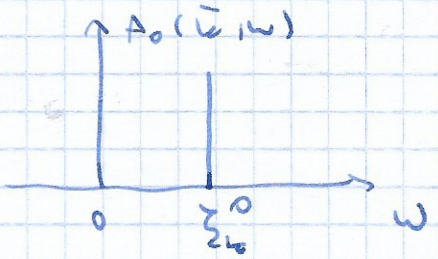
$\rightarrow$ Fermi gas: $\quad G_0^R(\vec{k},\omega) = \dfrac{1}{\omega - \underbrace{\varepsilon_{\vec{k}}^0 + \mu}_{\xi_{\vec{k}}^0 \equiv \varepsilon_{\vec{k}}^0 - \mu} + i\eta}$

$$G_0^R(\vec{k},t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{e^{-i\omega t}}{\omega + i\eta - \xi_{\vec{k}}^0} =$$

$$= -i\,\Theta(t)\, e^{-i\xi_{\vec{k}}^0 t}$$

Spectral function $\quad A_0(\vec{k},\omega) = -\frac{1}{\pi} \operatorname{Im} G_0^R(\vec{k},\omega) = \delta(\omega - \xi_{\vec{k}}^0)$





$\rightarrow$ Fermi liquid: $\quad \boxed{G_R(\vec{k},\omega) = \dfrac{1}{\omega + i\eta - \xi_{\vec{k}}^0 - \Sigma^R(\vec{k},\omega)}}$

Self-energy

Spectral function

$$\boxed{A(\vec{k},\omega) = -\frac{1}{\pi} \frac{\operatorname{Im}\Sigma^R(\vec{k},\omega)}{(\omega - \xi_{\vec{k}}^0 - \operatorname{Re}\Sigma^R(\vec{k},\omega))^2 + (\operatorname{Im}\Sigma^R(\vec{k},\omega))^2}}$$

When $\operatorname{Im}\Sigma^R(\vec{k},\omega) \rightarrow 0$ it reduces to the δ-function

If $\operatorname{Im}\Sigma^R(\vec{k},\omega)$ varies weakly for $\omega \simeq \xi_{\vec{k}}$ the maximum (peak) of spectral function is determined by

$$\boxed{\xi_{\vec{k}} - \xi_{\vec{k}}^0 - \operatorname{Re}\Sigma^R(\vec{k},\xi_{\vec{k}}) = 0}$$

using $\left.\frac{\partial}{\partial\omega} \operatorname{Im}\Sigma^R(\vec{k},\omega)\right|_{\omega=\xi_{\vec{k}}} = 0$

energy of qp with momentum $\vec{k}$

E.g. at the Fermi momentum k_F , $\xi_{k_F} = 0$

$$\rightarrow \quad \xi^0_{k_F} + \Sigma^R(k_F, 0) = 0$$

$\boxed{\text{since } \Sigma^R(k_F,0) \in \mathbb{R} \text{ at } T=0}$

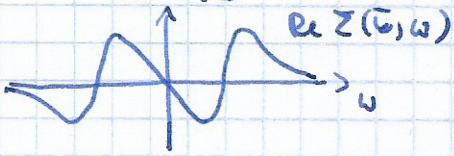
in interacting system $\mu \neq \varepsilon_F^0 = \frac{k_F^2}{2m}$ so $\xi^0_{k_F} \neq 0$.
But $k_F = (3\pi^2 n)^{1/3}$ – Luttinger theorem.

For ω very close to $\xi_{\vec k}$:

$$\omega - \xi^0_{\vec k} - \mathrm{Re}\,\Sigma^R(\vec k, \omega) \simeq \omega \overbrace{- \xi^0_{\vec k} - \mathrm{Re}\,\Sigma^R(\vec k, \xi_{\vec k})}^{-\xi_{\vec k}} -$$

$$- (\omega - \xi_{\vec k}) \frac{\partial}{\partial \omega} \mathrm{Re}\,\Sigma^R(\vec k, \omega)\Big|_{\omega = \xi_{\vec k}} =$$

$$= \frac{\omega - \xi_{\vec k}}{Z_{\vec k}}$$



with

$$\boxed{Z_{\vec k} = \frac{1}{1 - \frac{\partial}{\partial \omega} \mathrm{Re}\,\Sigma^R(\vec k, \omega)\Big|_{\omega = \xi_{\vec k}}}}$$

$\boxed{\text{quasiparticle weight } 0 < Z_{\vec k} \leq 1}$

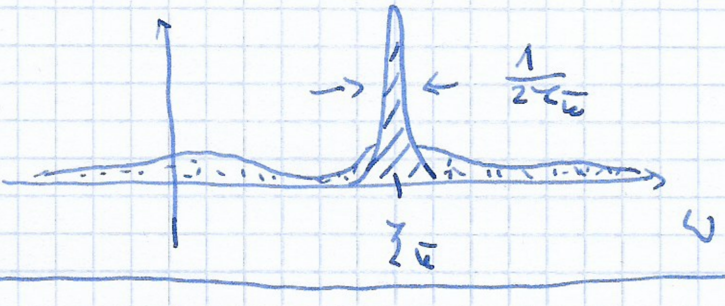
Fermi liquid renormalization function.

Hence,

$$\boxed{A(\vec k, \omega) = \frac{Z_{\vec k}}{\pi} \frac{\frac{1}{2\tau_{\vec k}}}{(\omega - \xi_{\vec k})^2 + \left(\frac{1}{2\tau_{\vec k}}\right)^2} + A_{inc}(\vec k, \omega)}$$

$$\boxed{\frac{1}{\tau_{\vec k}} = -2 Z_{\vec k} \,\mathrm{Im}\,\Sigma^R(\vec k, \xi_{\vec k})}$$ inverse of quasiparticle life time

$$\boxed{\shortparallel \atop \Gamma(\vec k)}$$



$$\langle \Psi^N | c^\dagger_{\vec k} | FS \rangle = \langle \Psi^N | \Psi_{qp} \rangle = Z_{\vec k}$$

Also, we find

$$\boxed{\begin{aligned} G^R(\vec{k},\omega) &= \int_{-\infty}^{\infty} d\omega' \frac{A(\vec{k},\omega')}{\omega + i\eta - \omega'} = \frac{z_{\vec{k}}}{\omega - \xi_{\vec{k}} + \frac{i}{2\tau_{\vec{k}}}} + G^R_{inc}(\vec{k},\omega) \\ G^R(\vec{k},\omega) &= -i\, z_{\vec{k}}\, \theta(t)\, e^{-i\xi_{\vec{k}} t - \frac{t}{2\tau_{\vec{k}}}} + G^R_{inc}(\vec{k},t) \end{aligned}}$$

there:

$$\frac{1}{\xi_{\vec{k}}} \ll t \ll \tau_{\vec{k}} = \frac{1}{\Gamma(\vec{k})}$$

<u>Momentum distribution</u>

$$n_{\vec{k}} = \langle c^+(\vec{k})\, c(\vec{k}) \rangle = G(\vec{k}, \tau = 0^-) = \frac{1}{\beta}\sum_{\omega_n} e^{i\omega_n \eta} G(\vec{k}, i\omega_n) =$$

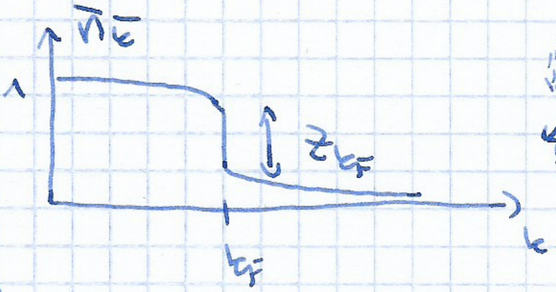
$$= \frac{1}{\beta}\sum_{\omega_n} e^{i\omega_n\eta} \int_{-\infty}^{\infty} d\omega \frac{A(\vec{k},\omega)}{i\omega_n - \omega} =$$

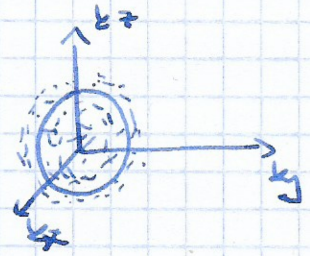
$$= \int_{-\infty}^{\infty} d\omega\, n_F(\omega)\, A(\vec{k},\omega) \underset{T=0}{=} \int_{-\infty}^{0} d\omega\, A(\vec{k},\omega)$$

when $\frac{1}{\tau_{\vec{k}}} \ll \xi_{\vec{k}} \rightarrow A(\vec{k},\omega) \simeq z_{k_F}\, \delta(\omega - \xi_{\vec{k}})$ for $\xi_{\vec{k}} \to 0$

$$\lim_{|\vec{k}|\to k_F^+} n_{\vec{k}} - \lim_{|\vec{k}|\to k_F^-} n_{\vec{k}} = \left[\lim_{|\vec{k}|\to k_F^+} - \lim_{|\vec{k}|\to k_F^-}\right] \int_{-\infty}^{0} d\omega\, z_{\vec{k}}\, \delta(\omega - \xi_{\vec{k}}) =$$

$$= -z_{k_F}$$





A sharp step at T=0 even in interacting fermions if $z_{k_F} \neq 0$

<u>Fermi liquid condition</u> $\rightarrow$ $\boxed{\text{existence of the Fermi surface}}$

§ 8. THERMODYNAMIC POTENTIAL AND LANDAU FUNCTIONS

Partition function of quasiparticles

$$\boxed{Z[h] = \int D[\bar\psi^* \bar\psi]\, e^{-S[\bar\psi^* \bar\psi] + \sum_{\bar k \sigma} h_{\bar k \sigma} \int_0^\beta d\tau\, \hat n_{\bar k \sigma}(\tau)}}$$

quasiparticle number $\hat n_{\bar k\sigma}(\tau) = \bar\psi^*_{\bar k\sigma}(\tau)\, \bar\psi_{\bar k\sigma}(\tau)$

quasiparticle field operators

quasiparticle occupation number

$$\boxed{n_{\bar k\sigma} = \langle \hat n_{\bar k\sigma}(\tau) \rangle = \frac{1}{\beta} \frac{\delta \ln Z[h]}{\delta h_{\bar k\sigma}}} \qquad (*)$$

Legendre transform → grand canonical potential

$$\Omega[n] = -\frac{1}{\beta} \ln Z[h] + \sum_{\bar k\sigma} h_{\bar k\sigma} n_{\bar k\sigma},$$

where $h_{\bar k\sigma} = h_{\bar k\sigma}[n]$ is obtained from $(*)$.

Stationary condition

$$\boxed{\frac{\delta \Omega[n]}{\delta n_{\bar k\sigma}} = h_{\bar k\sigma}} \quad \text{– equation of states}$$

and at equilibrium $h = 0$.

For FL we need a variation $\delta\Omega$ when $\bar n = n|_{h=0}$ ~~the~~ is changed by δn

Expanding

$\delta\Omega[\delta n] = \Omega[n+\delta n] - \Omega[n]$ to second order

$$\delta\Omega[\delta n] = \frac{1}{2}\sum_{\bar{k}\bar{k}'}\sum_{\sigma\sigma'} \left.\frac{\delta^2\Omega[n]}{\delta n_{\bar{k}\sigma}\,\delta n_{\bar{k}'\sigma'}}\right|_{n=\bar{n}} \delta n_{\bar{k}\sigma}\,\delta n_{\bar{k}'\sigma'}$$

Taking a functional derivative of the equation of states and using (x) we find

$$\frac{1}{\beta}\sum_{\bar{k}_3\sigma_3}\left(\frac{\delta^2\Omega[n]}{\delta n_{\bar{k}_1\sigma_1}\,\delta n_{\bar{k}_3\sigma_3}}\right)\cdot\left(\frac{\delta^2\ln Z[h]}{\delta h_{\bar{k}_3\sigma_3}\,\delta h_{\bar{k}_2\sigma_2}}\right) = \delta_{\bar{k}_1\bar{k}_2}\delta_{\sigma_1\sigma_2}$$

Hence,

$$\delta\Omega[\delta n] = \frac{1}{2}\sum_{\substack{\bar{k}\bar{k}'\\ \sigma\sigma'}} \bar{\chi}^{-1}_{\sigma\sigma'}(\bar{k},\bar{k}')\,\delta n_{\bar{k}\sigma}\,\delta n_{\bar{k}'\sigma'}$$

where

$$\boxed{\bar{\chi}_{\sigma\sigma'}(\bar{k},\bar{k}') \equiv \frac{1}{\beta}\left.\frac{\delta^2\ln Z[h]}{\delta h_{\bar{k}\sigma}\,\delta h_{\bar{k}'\sigma'}}\right|_{h=0} = \frac{1}{\beta}\int_0^\beta d\tau\,d\tau'\,\langle n_{\bar{k}\sigma}(\tau)\,n_{\bar{k}'\sigma'}(\tau')\rangle}$$

linear response function to the external field h.

Using our earlier result on FL, page 12, we obtain

$$\boxed{\frac{1}{V}f_{\sigma\sigma'}(\bar{k},\bar{k}') = \frac{\delta_{\sigma\sigma'}\delta_{\bar{k}\bar{k}'}}{n_F'(\xi_{\bar{k}})} + \bar{\chi}^{-1}_{\sigma\sigma'}(\bar{k},\bar{k}')}$$

The Landau f function is related to the response function.

Relation between f and particle-hole vertex Γ_{ph}

Introduce $\qquad k = (i\omega_n, \bar{k})$ etc.

$$\chi_{\sigma\sigma'}(k,k';q) = \langle \bar{\psi}^*_\sigma(k)\bar{\psi}_\sigma(k+q)\bar{\psi}^*_{\sigma'}(k'+q)\bar{\psi}_{\sigma'}(k')\rangle -$$
$$- \langle \bar{\psi}^*_\sigma(k)\bar{\psi}_\sigma(k+q)\rangle\langle\bar{\psi}^*_{\sigma'}(k'+q)\bar{\psi}_{\sigma'}(k')\rangle$$

and

$$\overline{\chi}_{\sigma\sigma'}(\bar{k},\bar{k}';q) = \frac{1}{\beta}\sum_{\omega_n\omega_n'} \chi_{\sigma\sigma'}(k,k';q)$$

Landau f function is related with

$\overline{\chi}_{\sigma\sigma'}(\bar{k},\bar{k}';q)$ in the $q \to 0$ limit.

However, $\bar{q} \to 0$, $i\omega_\nu \to$ limits do not commute!

Thus the external field $h_{\bar{k}\sigma}$ to modify the quasiparticle distribution function is understood as

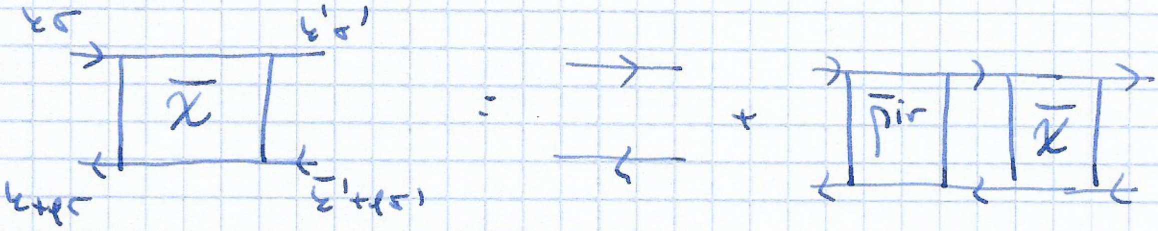
$$h_{\bar{k}\sigma}\int_0^\beta d\tau\, \psi^*_{\bar{k}\sigma}(\tau)\psi_{\bar{k}\sigma}(\tau) = h_{\bar{k}\sigma}\sum_{\omega_n} \bar{\psi}^*_\sigma(k)\bar{\psi}_\sigma(k) =$$
$$= h_{\bar{k}\sigma} \lim_{\bar{q}\to 0}\left[\lim_{\omega_\nu\to 0}\sum_{\omega_n} \bar{\psi}^*_\sigma(k)\bar{\psi}_\sigma(k+q)\right]$$

Hence, we define

$$\overline{\chi}_{\sigma\sigma'}(\bar{k},\bar{k}') = \lim_{\bar{q}\to 0}\left[\lim_{\omega_\nu\to 0} \overline{\chi}_{\sigma\sigma'}(\bar{k},\bar{k}';q)\right]$$

$$\boxed{\begin{array}{l} (i\omega_\nu \to 0,\ \bar{q}\to 0) - \omega\text{-limit} \\ (\bar{q}\to 0,\ i\omega_\nu \to 0) - \bar{q}\text{-limit} \end{array}}$$

Two-particle Green's function $\bar{\chi}_{\sigma\sigma'}(k,k';q)$ satisfies the <u>Bethe-Salpeter equation</u>



P50

$$\boxed{\begin{aligned}\bar{\chi}_{\sigma\sigma'}(kk';q) &= \bar{\Pi}_{\sigma\sigma'}(kk';q) - \\ &- \frac{1}{\beta}\frac{1}{V}\sum_{\substack{k_1k_2\\ \sigma_1\sigma_2}} \bar{\Pi}_{\sigma\sigma_1}(kk_1;q)\,\bar{\Gamma}^{irr}_{\sigma_1\sigma_2}(k_1k_2;q)\,\chi_{\sigma_2\sigma'}(k_2k';q)\end{aligned}}$$

where

$$\bar{\Gamma}^{irr}_{\sigma\sigma'}(kk';q) = \bar{\Gamma}^{irr}_{ph,\sigma\sigma\sigma'\sigma'}(k+q,k;k',k'+q)$$

is the <u>irreducible (2PI) vertex in the p-h channel</u>,

and

$$\bar{\Pi}_{\sigma\sigma'}(kk';q) = -\delta_{\sigma\sigma'}\delta_{kk'}\,\bar{G}(k)\,\bar{G}(k+q)$$

is the <u>quasi-particle – quasi-hole propagator</u>.

Using a quasiparticle propagator

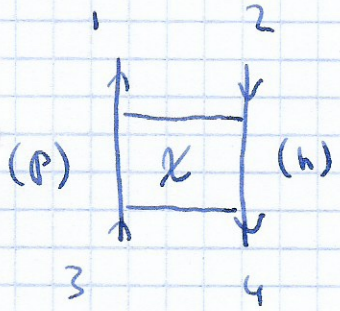
$$\bar{G}(k,i\omega_n) = \frac{1}{i\omega_n - \xi_{\bar{k}}}$$

with

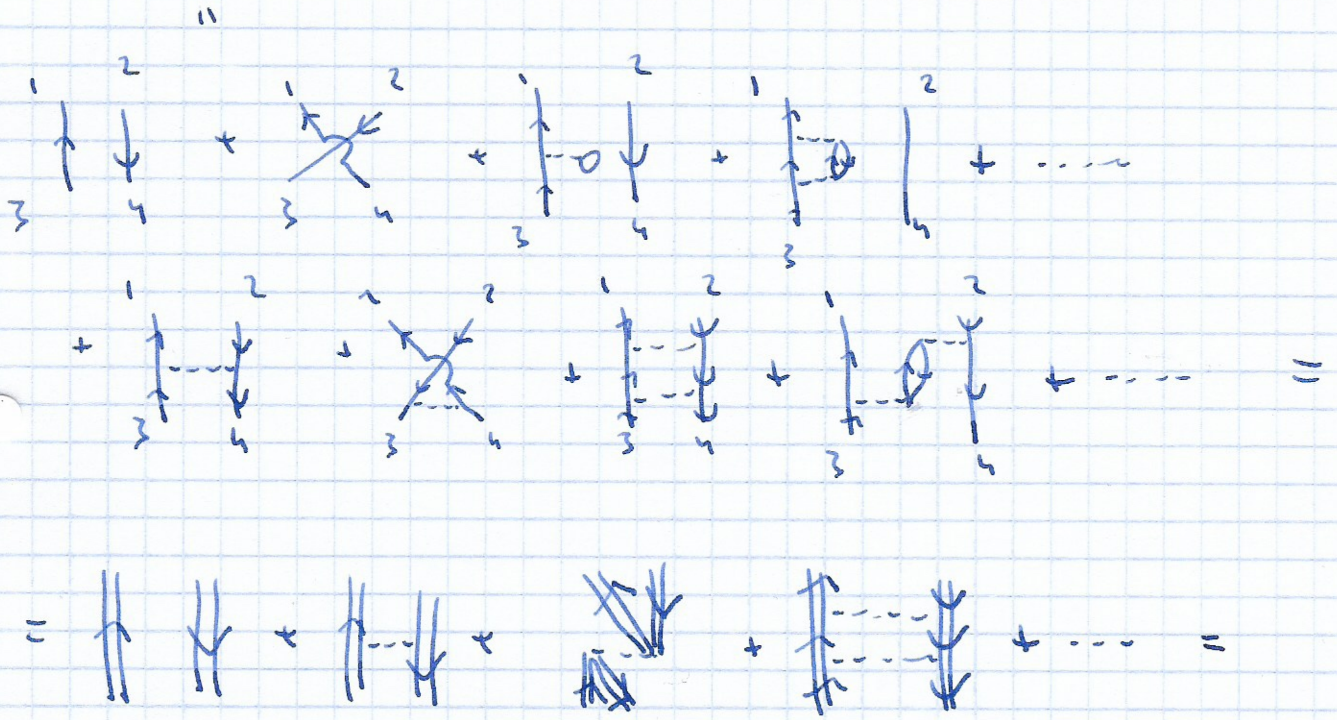
$$G(\bar{k},i\omega_n) = Z_{\bar{k}}\,\bar{G}(\bar{k},i\omega_n) + G_{inc}(\bar{k},i\omega_n)$$

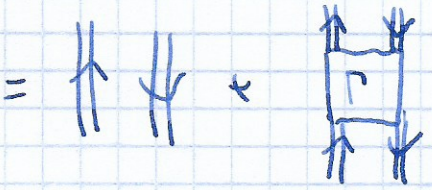
$$Z_{\bar{k}} = \left(1 - \left.\frac{\partial\Sigma(\bar{k},z)}{\partial z}\right|_{z=0}\right)^{-1}$$

$$\xi_{\bar{k}} = Z_{\bar{k}}\left[\xi^0_{\bar{k}} + \Sigma(\bar{k},0)\right]$$

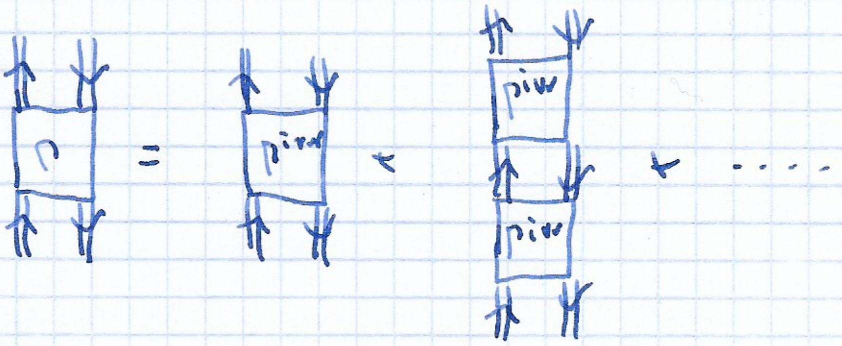


$$= \langle \Psi | \psi^{+}(1)\, \psi(2)\, \psi^{+}(3)\, \psi(4) | \Psi \rangle$$

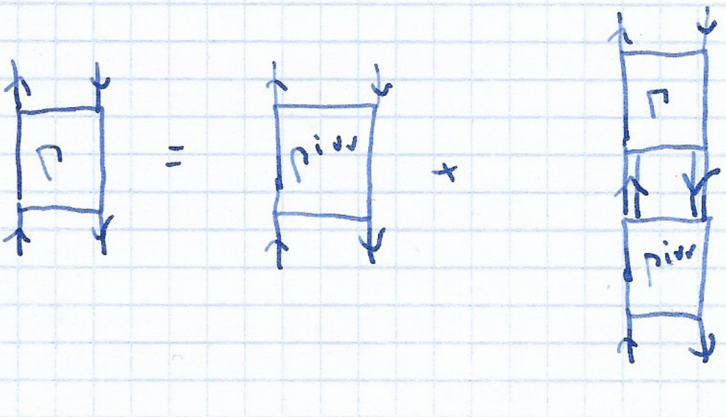




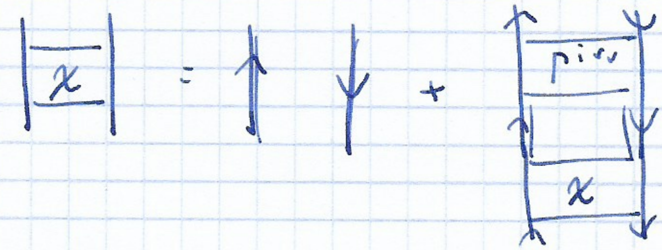
Γ - full vertex (scattering) function

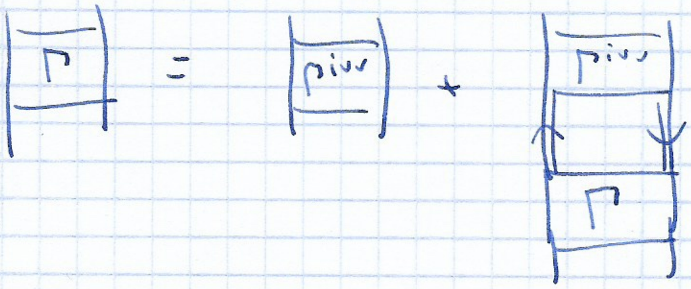


Γ^{irr} - irreducible vertex function (not split by cutting 2 fermion lines)



Bethe - Salpeter eq.


x
=
+
piv
x


Γ
=
piv
+
piv
Γ

We consider

$$\bar{\Pi}_{\sigma\sigma'}(\bar{k},\bar{k}';q) = \frac{1}{\beta}\sum_{\omega_n\omega_{n'}} \Pi_{\sigma\sigma'}(k,k';q) =$$

$$= -\delta_{\sigma\sigma'}\delta_{kk'}\frac{1}{\beta}\sum_{\omega_n} G(k)\,G(k+q) =$$

$$= \delta_{\sigma\sigma'}\delta_{kk'}\frac{n_F(\xi_{\bar{k}+\bar{q}}) - n_F(\xi_{\bar{k}})}{i\omega_\nu - \xi_{\bar{k}+\bar{q}} + \xi_{\bar{k}}} \underset{T\to 0}{\overset{\bar{q}\to 0}{\simeq}}$$

$$\simeq -\delta_{\sigma\sigma'}\delta_{kk'}\frac{\bar{v}_F^* \cdot \bar{q}}{i\omega_\nu - \bar{v}_F^* \cdot \bar{q}}\,\delta(\xi_{\bar{k}}) \underset{q\to 0}{=}$$

$$= \begin{cases} 0 & \text{if } \dfrac{|\bar{q}|}{\omega_\nu} \to 0 \\ \delta_{\sigma\sigma'}\delta_{kk'}\delta(\xi_{\bar{k}}) & \text{if } \dfrac{\omega_\nu}{|\bar{q}|} \to 0 \end{cases}$$

Since Γ^{irr} is regular in $q\to 0$ we take

$$\Gamma^{irr}_{\sigma\sigma'}(\bar{k}_F, \bar{k}_F') \qquad \bar{k}_F = k_F\hat{k}\,,\ \bar{k}_F' = k_F\hat{k}'$$

Hence we get

$$\bar{\chi}_{\sigma\sigma'}(\bar{k},\bar{k}';q) = \bar{\Pi}_{\sigma\sigma'}(\bar{k},\bar{k}';q) -$$

$$-\frac{1}{V}\sum_{\substack{\bar{k}_1\bar{k}_2\\ \sigma_1\sigma_2}} \bar{\Pi}_{\sigma\sigma_1}(\bar{k},\bar{k}_1;q)\,\Gamma^{irr}_{\sigma_1\sigma_2}(\bar{k}_{F1},\bar{k}_{F2})\,\bar{\chi}_{\sigma_2\sigma}(\bar{k}_2,\bar{k}';q)$$

or in a matrix form

$$\bar{\chi} = \bar{\Pi} - \frac{1}{V}\bar{\Pi}\,\Gamma^{irr}\,\bar{\chi} \;\rightarrow\; \bar{\chi}^{-1} = \bar{\Pi}^{-1} + \frac{1}{V}\Gamma^{irr}$$

In the $\bar{q} \to 0$ limit

$$\bar{\chi}^{-1}_{\sigma\sigma'}(\bar{k},\bar{k}') = -\frac{\delta_{\sigma\sigma'}\delta_{\bar{k}\bar{k}'}}{n_F'(\beta\xi)} + \frac{1}{V}\Gamma^{irr}_{\sigma\sigma'}(\bar{k}_F,\bar{k}_F')$$

Hence

$$\boxed{f_{\sigma\sigma'}(\bar{k},\bar{k}') = \bar{\Gamma}^{irr}_{\sigma\sigma'}(\bar{k}_F,\bar{k}_F')}$$

Now, we relate $\bar{\Gamma}^{irr}$ to the full (1PI) vertex $\bar{\Gamma} = \bar{\Gamma}_{ph}$.

It satisfying the Bethe-Salpeter equation

$$\bar{\Gamma}_{\sigma\sigma'}(k,k';q) = \bar{\Gamma}^{irr}_{\sigma\sigma'}(k,k') -$$

$$-\frac{1}{\beta V}\sum_{\substack{k_1k_2\\ \sigma_1\sigma_2}} \bar{\Gamma}^{irr}_{\sigma_1\sigma_2}(k,k_1)\bar{\Pi}_{\sigma_1\sigma_2}(k_1k_2;q)\bar{\Gamma}_{\sigma_2\sigma'}(k_2k';q)$$

we put $\bar{q}=0$ limit

neglecting frequency dependence of $\bar{\Gamma}$ and $\bar{\Gamma}^{irr}$, we can carry frequency sum

$$\bar{\Gamma}_{\sigma\sigma'}(\bar{k},\bar{k}';q) = \bar{\Gamma}^{irr}_{\sigma\sigma'}(\bar{k},\bar{k}') -$$

$$-\frac{1}{V}\sum_{\substack{\bar{k}_1\bar{k}_2\\ \sigma_1\sigma_2}} \bar{\Gamma}^{irr}_{\sigma_1\sigma_2}(\bar{k},\bar{k}_1)\bar{\Pi}_{\sigma_1\sigma_2}(\bar{k}_1,\bar{k}_2;q)\bar{\Gamma}_{\sigma_2\sigma'}(\bar{k}_2,\bar{k}';q)$$

$\bar{\Pi}$ is vanishing in ω-limit

$$\bar{\Gamma}^{irr}_{\sigma\sigma'}(\bar{k}_F,\bar{k}_F') = \lim_{\omega\to 0}\left[\lim_{\bar{q}\to 0}\bar{\Gamma}_{\sigma\sigma'}(\bar{k}_F,\bar{k}_F';q)\right] \equiv \bar{\Gamma}^{\omega}_{\sigma\sigma'}(\bar{k}_F,\bar{k}_F')$$

Hence,

$$\boxed{f_{\sigma\sigma'}(\bar{k}, \bar{k}') = \bar{\Gamma}^{\omega}_{\sigma\sigma'}(\bar{k}_F, \bar{k}_F')}$$

Finally, we need to relate $\overbrace{\bar{\Gamma}^{\omega}}^{\text{quasiparticle vertex}}$ to the particle vertex Γ.

We define in terms of bare fermions:

$$\chi_{\sigma\sigma'}(kk';q) = \langle \psi^{\dagger}_{\sigma}(k)\psi_{\sigma}(k+q)\psi^{\dagger}_{\sigma'}(k'+q)\psi_{\sigma'}(k')\rangle - \langle \psi^{\dagger}_{\sigma}(k)\psi_{\sigma}(k+q)\rangle\langle \psi^{\dagger}_{\sigma'}(k'+q)\psi_{\sigma'}(k')\rangle$$

which satisfies

$$\chi_{\sigma\sigma'}(kk';q) = \Pi_{\sigma}(kk';q) - \frac{1}{\beta^2}\frac{1}{V}\sum_{\substack{k_1k_2\\ \sigma_1\sigma_2}} \Pi_{\sigma\sigma_1}(kk_1;q)\,\Gamma_{\sigma_1\sigma_2}(k_1k_2;q)\,\Pi_{\sigma_2\sigma'}(k_2k';q)$$

where

$$\Pi_{\sigma\sigma'}(k,k';q) = -\delta_{\sigma\sigma'}\delta_{kk'}\,G(k)G(k+q) = -\delta_{\sigma\sigma'}\delta_{kk'}\Big[\underbrace{Z^2\,\bar{G}(k)\bar{G}(k+q)}_{\text{coherent part}} + \varphi(k)\Big]$$

non-singular in $q \to 0$

in matrix notation:

$$\chi|_{coh} = Z^2\bar{\Pi} - Z^2\bar{\Pi}\,\Gamma\,Z^2\bar{\Pi}$$

Since $\bar{\chi} = \bar{\Pi} - \bar{\Pi}\bar{\Gamma}\bar{\Pi}$ we conclude $\chi|_{coh} = Z^2\bar{\chi}$ with $\bar{\Gamma} = Z^2\Gamma$

This proves

$$\boxed{f_{\sigma\sigma'}(\bar{k},\bar{k}') = Z^2\,\Gamma^{\omega}_{\sigma\sigma'}(\bar{k}_F, \bar{k}_F')} \quad !$$

§ 8. QUANTUM BOLTZMANN EQUATION

Consider quasi-particle Wigner distribution function

$$\boxed{n_{\bar{k}\sigma}(\bar{r},\tau) = \int d_3 r' e^{-i\bar{k}\cdot\bar{r}'} \langle \bar{\Psi}^*_\sigma(\bar{r}-\tfrac{\bar{r}'}{2},\tau)\, \bar{\Psi}_\sigma(\bar{r}+\tfrac{\bar{r}'}{2},\tau)\rangle}$$

and its Fourier transform

$$\boxed{\begin{aligned} n_{\bar{k}\sigma}(\bar{q},\omega_\nu) &= \frac{1}{\beta V}\int_0^\beta d\tau \int d_3 r\, e^{-i(\bar{q}\cdot\bar{r}-\omega_\nu\tau)}\, n_{\bar{k}\sigma}(\bar{r},\tau) = \\ &= \frac{1}{\beta}\sum_{\omega_n} \langle \bar{\Psi}^*_\sigma(k)\, \bar{\Psi}_\sigma(k+q)\rangle \end{aligned}}$$

This Wigner function is a quantum analog of the semiclassical distribution function $n_{\bar{k}\sigma}(\bar{r},t)$. It is not positive defined but its moments are ok.

Consider a system in the presence of a source field $h_{\bar{k}\sigma}(q) = h^*_{\bar{k}\sigma}(-q)$ coupled to the quasi-particle operator

$$\hat{n}_{\bar{k}\sigma}(q) = \frac{1}{\beta}\sum_{\omega_n} \bar{\Psi}^*_\sigma(k)\,\Psi_\sigma(k+q)$$

$$S_h = -\beta \sum_{\bar{k}\sigma}\sum_{q} h_{\bar{k}\sigma}(-q)\,\hat{n}_{\bar{k}\sigma}(q)$$

Then the Wigner function is

$$\boxed{n_{\bar{k}\sigma}(q) = \langle \hat{n}_{\bar{k}\sigma}(q)\rangle = \frac{1}{\beta}\frac{\delta \ln Z[h]}{\delta h_{\bar{k}\sigma}(-q)}}$$

We introduce a functional of the Wigner distribution function –

– analog of the grand canonical potential

$$\boxed{\Omega[n] = -\frac{1}{\beta}\ln Z[h] + \sum_{\bar{k}\sigma}\sum_{\bar{q}} h_{\bar{k}\sigma}(-\bar{q})\, n_{\bar{k}\sigma}(\bar{q})}$$

The equation of states is

$$\boxed{\frac{\delta\Omega[n]}{\delta n_{\bar{k}\sigma}(\bar{q})} = h_{\bar{k}\sigma}(-\bar{q})}$$

For free fermions → computed exactly.

Close to equilibrium we take small fluctuations

$$\boxed{n_{\bar{k}\sigma}(\bar{q}) = \delta_{\bar{q},0}\,\bar{n}_{\bar{k}} + \delta n_{\bar{k}\sigma}(\bar{q})}$$

To lowest order in δn

$$\delta\Omega[\delta n] = \frac{1}{2}\sum_{\substack{\bar{k}\bar{k}'\\ \sigma\sigma'}}\sum_{\bar{q}\bar{q}'} \underbrace{\left.\frac{\delta^{(2)}\Omega[n]}{\delta n_{\bar{k}\sigma}(-\bar{q})\,\delta n_{\bar{k}'\sigma'}(\bar{q}')}\right|_{n=\bar{n}}}_{\delta_{\bar{q}\bar{q}'}\,\mathcal{X}^{-1}_{\sigma\sigma'}(\bar{k},\bar{k}';\bar{q})} \delta n_{\bar{k}\sigma}(-\bar{q})\,\delta n_{\bar{k}'\sigma'}(\bar{q}')$$

$$= \frac{1}{2}\sum_{\substack{\bar{k}\bar{k}'\\ \sigma\sigma'}}\sum_{\bar{q}}\left\{\frac{\delta_{\sigma\sigma'}\delta_{\bar{k}\bar{k}'}}{n_F'(\xi_{\bar{k}})}\,\frac{i\omega_\nu - \bar{v}^*_{\bar{k}}\cdot\bar{q}}{\bar{v}^*_{\bar{k}}\cdot\bar{q}} + \frac{1}{V} f_{\sigma\sigma'}(\bar{k},\bar{k}')\right\}\delta n_{\bar{k}\sigma}(-\bar{q})\,\delta n_{\bar{k}'\sigma'}(\bar{q})$$

In the absence of a source field $h_{\bar k\sigma}(q) = 0$ it gives the quantum Boltzmann equation

$$\boxed{0 = (i\omega_\nu - \bar v^*_{\bar k}\cdot\bar q)\,\delta n_{\bar k\sigma}(q) - \delta(\xi_{\bar k})\,\bar v^*_{\bar k}\cdot\bar q\,\frac{1}{V}\sum_{\bar k'\sigma'} f_{\sigma\sigma'}(\bar k,\bar k')\,\delta n_{\bar k'\sigma'}(q)}$$

Satisfied by the Wigner distribution function $n_{\bar k\sigma}(q)$.

To get the real time dynamics we need analytical continuation

$$i\omega_\nu \longrightarrow \omega + i0^+$$

It's solution is written

$$\delta n_{\bar k\sigma}(q) = v_F^*\,\delta(\xi_{\bar k})\,u_\sigma(\hat k, q)$$

displacement of FS in $\hat k$ direction

Hence

$$\boxed{\begin{aligned}\delta\Omega[u] = \frac{V}{2} N^*(0)\, v_F^{*2} \sum_{q}\sum_{\sigma\sigma'} \Big\{ &\delta_{\sigma\sigma'} \int \frac{d\Omega_{\hat k}}{4\pi}\, \frac{v_F^*\hat k\cdot\bar q}{v_F^*\hat k\cdot\bar q - i\omega_\nu}\, u_\sigma(\hat k,-q)\,u_\sigma(\hat k,q) \\ &+ \frac{1}{2}\int \frac{d\Omega_{\hat k}}{4\pi}\,\frac{d\Omega_{\hat k'}}{4\pi}\, F_{\sigma\sigma'}(\bar k_F,\bar k'_F)\, u_\sigma(\hat k,-q)\, u_{\sigma'}(\hat k',q)\Big\}\end{aligned}}$$

→ static properties, thermodynamics of FS

→ dynamic properties of FL (collective modes, response functions)