§10. LUTTINGER-WARD FUNCTIONAL AND LUTTINGER THEOREM

Formalizm

$$Z = \mathrm{Tr}\, e^{-\beta \hat{K}} \qquad \hat{K} = \hat{H} - \mu \hat{N}$$

$$G_{xx'} = \langle T \hat{\psi}_x \hat{\psi}^+_{x'} \rangle \qquad x = (\vec{r}, \tau, \sigma) \text{ etc.}$$

$$\hat{K} = \hat{K}_0 + \hat{V} \qquad \sum_x = \frac{1}{\hbar}\int_0^{\hbar\beta} d\tau \int d\vec{r} \sum_\sigma$$

$\hat{V}$ – interaction

$$Z = Z_0 \left\langle T e^{-\frac{1}{\hbar}\int_0^{\hbar\beta} d\tau\, \hat{V}(\tau)} \right\rangle_0$$

Legendre transformation

A – source field

$$Z[A] = Z \left\langle T e^{\hat{\psi}^+_x A_{xx'} \hat{\psi}_{x'}} \right\rangle$$

$$\rightarrow \quad \left.\frac{\delta Z[A]}{\delta A_{xx'}}\right|_{A=0} = Z \langle T \hat{\psi}^+_x \hat{\psi}_{x'} \rangle =$$

$$= \pm Z \langle T \hat{\psi}_{x'} \hat{\psi}^+_x \rangle = \pm Z\, G_{x'x}$$

"Free energy"

$$F[A] = -\ln Z[A]$$

$$\delta F[A] = \mp G_{x'x}\, \delta A_{xx'}$$

"Entalpy" Legendre transf. $A \rightarrow G$

$$\mathcal{H}[G] = F[A] \pm G_{x'x} A_{xx'}$$

$$\delta \mathcal{H}[G] = \pm A_{x'x}\, \delta G_{xx'}$$

Free system

$$Z_0[A] = \left[\det(G_0^{-1} - A)\right]^{\mp 1} = e^{\mp \operatorname{Tr} \ln (G_0^{-1} - A)}$$

For $A = 0$

$$Z_0 = e^{\mp \operatorname{Tr} \ln G_0^{-1}} = e^{\mp \lim_{\tau \to 0^-} \sum_{\vec{k}} \sum_{\omega_n} e^{-i\omega_n \tau} \ln G_0^{-1}(\vec{k}, \omega_n)} =$$

$$= e^{\mp \sum_{\vec{k}} \ln(1 \mp e^{-\beta(\varepsilon_{\vec{k}} - \mu)})} = \prod_{\vec{k}} \left(1 \mp e^{-\beta(\varepsilon_{\vec{k}} - \mu)}\right)^{\mp 1}$$

$$G_0^{-1}(\vec{k}, \omega_n) = -i\hbar\omega_n + \varepsilon_{\vec{k}} - \mu$$

Free energy $\quad F_0[A] = \pm \operatorname{Tr} \ln (G_0^{-1} - A)$

$$G^{-1} = G_0^{-1} - A \rightarrow A = G_0^{-1} - G^{-1}$$

entalpy

$$\mathcal{H}_0[G] = \pm \operatorname{Tr}\left[-\ln G + [G_0^{-1} G - 1]\right]$$

$$\delta \mathcal{H}[G] = \mp \operatorname{Tr}\left[(G^{-1} - G_0^{-1})\, \delta G\right]$$

Interacting system

entalpy

$$\boxed{\begin{gathered} \mathcal{H}[G] = \mathcal{H}_0[G] - \underbrace{\Phi[G]}_{\text{Ward functional}} \\ \underbrace{\Sigma_{x'x}}_{\text{self-energy}} = \pm \frac{\delta \Phi[G]}{\delta G_{xx'}} \end{gathered}}$$

$$\rightarrow \quad \delta \Phi[G] = \pm \operatorname{Tr}[\Sigma\, \delta G] = \pm \Sigma_{x'x}\, \delta G_{xx'}$$

$$\delta\Omega[G] = \mp \mathrm{Tr}\left[\left(G^{-1} - G_0^{-1} + \Sigma\right)\delta G\right] = \pm \mathrm{Tr}\left[A\,\delta G\right]$$

when $A = 0$

$$\frac{\delta\Omega[G]}{\delta G_{x'x}} = \mp\left(G^{-1}_{x x'} - G^{-1}_{0,\,x x'} + \Sigma_{x x'}\right) = 0$$

$$\Rightarrow \quad \boxed{G^{-1}_{x x'} = G^{-1}_{0,\,x x'} - \Sigma_{x x'}}$$ Dyson equation

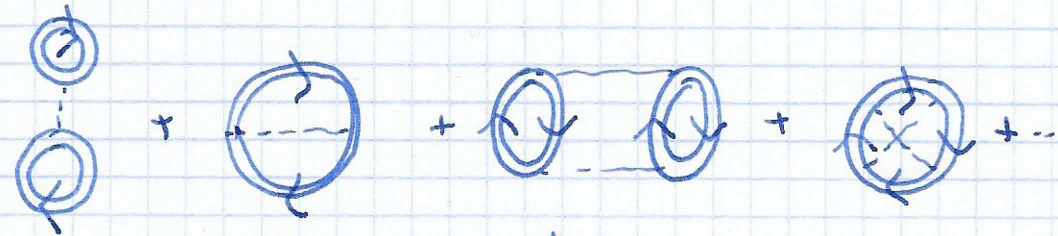
$$\Rightarrow \underbrace{\Omega(T,\mu)}_{\text{physical}} = -k_B T \ln Z = k_B T \ln F[0] = k_B T \;\mathrm{extr}\left(\Omega[G]\right)$$

extremum (saddle point) with respect to G

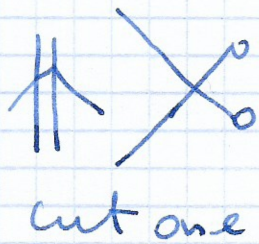
$Z[A] = Z_0 \{ 1 + \text{sum of all vacuum diagrams} \}$

$F[A] = F_0 - \{ \text{Sum of all connected vacuum diagrams} \}$

$\Phi[G] = \{$ sum of all vacuum diagrams with thick lines identified by G $\}$

$\Phi[G] =$ 

$\Sigma[G] = \dfrac{\delta\Phi[G]}{\delta G} =$ 

$\dfrac{1}{\delta G} =$ 

cut one ‖ line

Luttinger theorem

$n = \frac{N}{V}$ - density

$\underline{T=0}$

$$n = 2\int \frac{d^dk}{(2\pi)^d} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{i\omega\eta} G(\bar{k}, i\omega) =$$

$$= 2\int \frac{d^dk}{(2\pi)^d} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{i\omega\eta} \Big\{ \frac{\partial}{\partial i\omega} \ln\left(-i\omega_n + \xi_k + \Sigma(\bar{k}, i\omega)\right) +$$

$$+ \underbrace{G(\bar{k}, i\omega) \frac{\partial \Sigma(\bar{k}, i\omega)}{\partial(i\omega)}}_{I} \Big\}$$

we show that the I term vanishes

$\lim_{|\omega|\to\infty} G(\bar{k}, i\omega) \Rightarrow \frac{1}{i\omega}$, $\Sigma(\bar{k}, i\omega) = \text{const.}$

$$\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} G(\bar{k}, i\omega) \frac{\partial \Sigma(\bar{k}, i\omega)}{\partial(i\omega)} \overset{\text{by parts}}{=} -\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \Sigma(\bar{k}, i\omega) \frac{\partial G(\bar{k}, i\omega)}{\partial(i\omega)}$$

$\Sigma = \frac{\delta \Phi}{\delta G}$ and it is invariant if $i\omega \to i\omega + i\varepsilon$ in all lines.

$\boxed{\delta\Phi = \frac{\delta\Phi}{\delta i\varepsilon} \delta(i\varepsilon) = 0}$

$$0 = \int \frac{d^dk}{(2\pi)^d} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{\delta \Phi[G]}{\delta G(\bar{k}, i\omega)} \frac{\partial G(\bar{k}, i\omega)}{\partial(i\omega)} =$$

$$= \int \frac{d^dk}{(2\pi)^d} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \Sigma(\bar{k}, i\omega) \frac{\partial G(\bar{k}, i\omega)}{\partial(i\omega)}$$

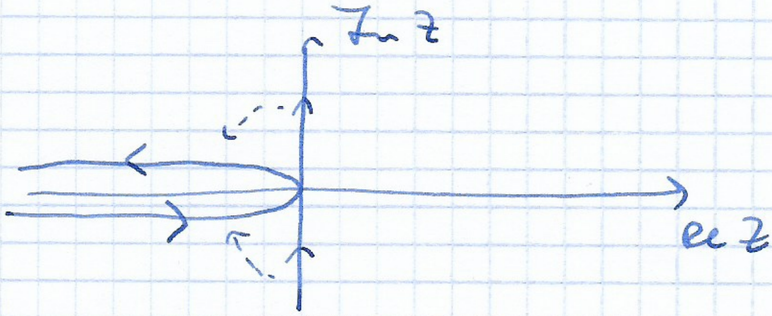
$\square$

Now,

$$n = -2\int \frac{d^dk}{(2\pi)^d} \int_{-i\infty}^{i\infty} \frac{dz}{2\pi i} e^{z\eta} \frac{\partial}{\partial z} \ln\left[-G(\bar{k}, z)\right]$$

$G(\bar{k}, z)$ analytic apart $z \in \mathbb{R}$

$e^{z\eta}$ enables us to choose a contour



$$n = -2\int \frac{d^d k}{(2\pi)^d}\left\{ \int_{-\infty}^{0} \frac{d\omega}{2\pi i}\, e^{\omega\eta}\, \frac{\partial}{\partial\omega} \ln\left[-G(\bar{k},\omega - i\eta)\right] + \right.$$

$$\left. + \int_{0}^{+\infty} \frac{d\omega}{2\pi i}\, e^{\omega\eta}\, \frac{\partial}{\partial\omega} \ln\left[-G(\bar{k},\omega + i\eta)\right]\right\} =$$

$$= -\frac{i}{\pi}\int \frac{d^d k}{(2\pi)^d} \int_{-\infty}^{0} d\omega\, \frac{\partial}{\partial\omega} \underbrace{\ln \frac{G^R(\bar{k},\omega)}{G^A(\bar{k},\omega)}}_{\varphi(\bar{k},\omega)} =$$

$\varphi(\bar{k},\omega)$ - phase (argument) of $G^R(\bar{k},\omega)$

$\varphi(\bar{k},-\infty) = -\pi$

$$= \frac{2}{\pi}\int \frac{d^d k}{(2\pi)^d}\left[\varphi(\bar{k},0) - \varphi(\bar{k},-\infty)\right] =$$

$$= \frac{2}{\pi}\int \frac{d^d k}{(2\pi)^d}\left[\varphi(\bar{k},0) + \pi\right]$$

since $\mathrm{Im}\,\Sigma(\bar{k},0) = 0$ and $\varphi(\bar{k},0) = \begin{cases} 0 \\ -\pi \end{cases}$ depending on $\mathrm{sgn}(G^R(\bar{k},0))$

hence

$$\boxed{n = 2\int \frac{d^d k}{(2\pi)^d}\, \Theta\left[G^R(\bar{k},0)\right]}$$

> For a given density n the volume of the Fermi surface in $\bar{k}$ space is the same for interacting and noninteracting system.