Linked cluster theorem

The average $\langle V(\mathcal{I})^k \rangle_0$ is separated onto connected contributions $\langle V(\mathcal{I})^k \rangle_c$ and products of them.

E.g.

$$\langle V(\mathcal{I}) \rangle = \langle V(\mathcal{I}) \rangle_c$$

$$\langle V(\mathcal{I})^2 \rangle = \langle V(\mathcal{I})^2 \rangle_c + \langle V(\mathcal{I}) \rangle_c^2$$

$$\langle V(\mathcal{I})^3 \rangle = \langle V(\mathcal{I})^3 \rangle_c + 3 \langle V(\mathcal{I})^2 \rangle_c \langle V(\mathcal{I}) \rangle_c + \langle V(\mathcal{I}) \rangle_c^3$$

$$\vdots$$

In general

$$\frac{1}{k!} \langle V(\mathcal{I})^k \rangle = \frac{1}{k!} \langle V(\mathcal{I})^k \rangle_c + \text{non-connected terms}$$

a non-connected term is in a form

$$\langle V(\mathcal{I})^{k_1} \rangle_c \langle V(\mathcal{I})^{k_2} \rangle_c \cdots \langle V(\mathcal{I})^{k_p} \rangle_c$$

$$k_1 + k_2 + \dots + k_p = k$$

with a combinatorial factor

$$\frac{1}{k!} \times \underbrace{\frac{k!}{k_1! k_2! \cdots k_p!}}_{\substack{\text{all possible ways} \\ \text{to group } k \text{ objects} \\ \text{in subsets of} \\ k_1 + k_2 + \dots + k_p \text{ objects,} \\ \text{when all } k_i \text{ are} \\ \text{distinct.}}} = \frac{1}{k_1! k_2! \cdots k_p!}$$

when m powers k_i are equal one must divide by $m!$ in addition.

$$\boxed{W(\lambda) = \ln Z(\lambda) = \ln Z(0) + \sum_k \frac{(-\lambda)^k}{k!} \langle V(\mathcal{I})^k \rangle_c}$$