Show that $\{\alpha_1' \ldots \alpha_N' | \alpha_1 \ldots \alpha_N\} = \xi^P \prod_\alpha n_\alpha!$ or zero

$$\{\alpha_1' \ldots \alpha_N' | \alpha_1 \ldots \alpha_N\} = N! \, (\alpha_1' \ldots \alpha_N' | \hat{P}_{(\xi)}^2 | \alpha_1 \ldots \alpha_N) =$$

← product state $|\,)$

$|\,\}$ symmetrized states

$$= N! \, (\alpha_1' \ldots \alpha_N' | \hat{P}_{(\xi)} | \alpha_1 \ldots \alpha_N) =$$

$$= \frac{N!}{N!} \sum_P \xi^P \langle \alpha_1' | \alpha_{P_1} \rangle \ldots \langle \alpha_N' | \alpha_{P_N} \rangle$$

$\{|\alpha\rangle\}$ - orthonormal $\Rightarrow$ $\langle \alpha_i' | \alpha_{P_i} \rangle \neq 0$ if $\alpha_i' = \alpha_{P_i} \; \forall i_N$

For fermions occupation (of $|\alpha_i\rangle$) is one, hence

$$\{\alpha_1' \ldots \alpha_N' | \alpha_1 \ldots \alpha_N\} = (-1)^P \qquad \begin{matrix} \{\alpha_1' \ldots \alpha_N'\} \\ \downarrow P \\ \{\alpha_1 \ldots \alpha_N\} \end{matrix}$$

For bosons $|\alpha_i\rangle$ occupied by n_i. We need a number of permutation $\{\alpha_1' \ldots \alpha_N'\}$ to $\{\alpha_1 \ldots \alpha_N\}$

$$\{\alpha_1' \ldots \alpha_N' | \alpha_1 \ldots \alpha_N\} = n_1! \ldots n_N!$$

Summary $\quad \{\alpha_1' \ldots \alpha_N' | \alpha_1 \ldots \alpha_N\} = \begin{cases} \xi^P \prod_\alpha (n_\alpha!) & \{\alpha_1' \ldots \alpha_N'\} \overset{P}{\Leftrightarrow} \{\alpha_1 \ldots \alpha_N\} \\ 0 & \text{otherwise} \end{cases}$

$n_\alpha = 0, 1, 2, \ldots$ - bosons

$n_\alpha = 0, 1$ - fermions $\qquad \sum_\alpha n_\alpha = N$

Symmetrized and normalized state

$$|\alpha_1 \ldots \alpha_N\rangle = \frac{1}{\sqrt{\prod_\alpha n_\alpha!}} |\alpha_1 \ldots \alpha_N\} \, .$$

e.g. $N=3$ $\qquad\qquad\qquad\qquad$ $\zeta=\pm 1$

$$\{\alpha_1'\alpha_2'\alpha_3'|\alpha_1\alpha_2\alpha_3\} = \Big(\langle\alpha_1'|\alpha_1\rangle\langle\alpha_2'|\alpha_2\rangle\langle\alpha_3'|\alpha_3\rangle \pm$$
$$\pm \langle\alpha_1'|\alpha_1\rangle\langle\alpha_2'|\alpha_3\rangle\langle\alpha_3'|\alpha_2\rangle \pm$$
$$\pm \langle\alpha_1'|\alpha_2\rangle\langle\alpha_2'|\alpha_1\rangle\langle\alpha_3'|\alpha_3\rangle +$$
$$+ \langle\alpha_1'|\alpha_3\rangle\langle\alpha_2'|\alpha_1\rangle\langle\alpha_3'|\alpha_2\rangle +$$
$$+ \langle\alpha_1'|\alpha_2\rangle\langle\alpha_2'|\alpha_3\rangle\langle\alpha_3'|\alpha_1\rangle \pm$$
$$\pm \langle\alpha_1'|\alpha_3\rangle\langle\alpha_2'|\alpha_2\rangle\langle\alpha_3'|\alpha_1\rangle \Big)$$

nonzero if $\{\alpha_1\alpha_2\alpha_3\}$ is a permutation of $\{\alpha_1'\alpha_2'\alpha_3'\}$

$(123) \Leftrightarrow (231)$

if, e.g. $\{123|456\} = 0$.

For fermion occupation is one, e.g. $\alpha_1' = \alpha_2$, $\alpha_2' = \alpha_1$, $\alpha_3' = \alpha_3$ $\qquad$ $N=3$, $\sum_i n_i = 3$

$$\{\alpha_1'\alpha_2'\alpha_3'|\alpha_1\alpha_2\alpha_3\} = -1$$

For bosons occupation is n_i, $\sum_i n_i = N$

e.g. $N=5$

$n_1 = 2$

$n_2 = 1$

$n_3 = 2$

$$\{\alpha_1'\alpha_1'\alpha_2'\alpha_3'\alpha_3'|\alpha_{p_1}\alpha_{p_2}\alpha_{p_3}\alpha_{p_4}\alpha_{p_5}\} = 2!\ 1!\ 2!$$

or $n_1 = 3$, $n_3 = 2$

$$\{\alpha_1'\alpha_1'\alpha_1'\alpha_3'\alpha_3'|\alpha_{p_1}\alpha_{p_2}\alpha_{p_3}\alpha_{p_4}\alpha_{p_5}\} = 3!\cdot 2!$$