Matsubara sum over functions with simple poles

Consider a fermionic sum

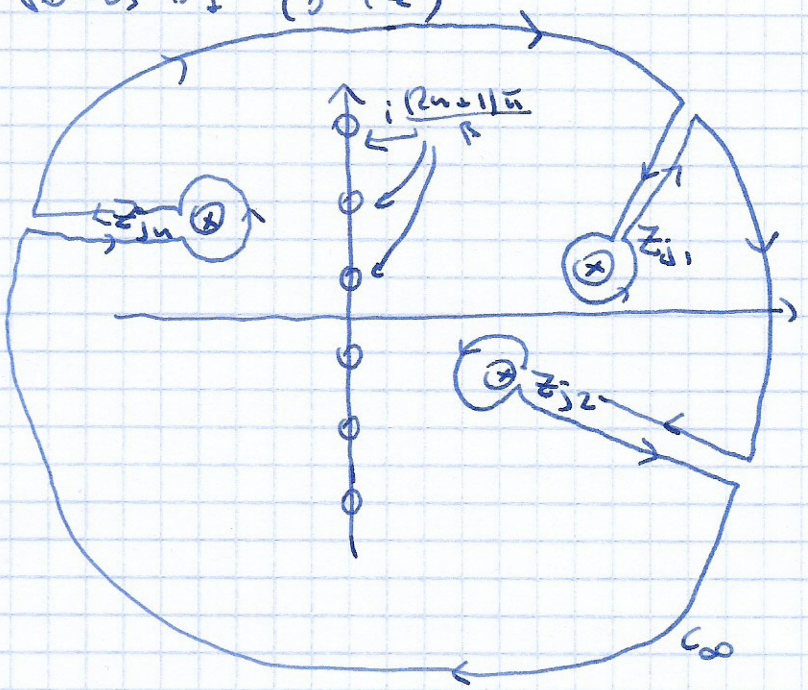
$$S_0^F(\tau) = \frac{1}{\beta} \sum_{ik_n} g_0(ik_n)\, e^{ik_n \tau} \qquad \tau > 0$$

$$k_n = \frac{(2n+1)\pi}{\beta}$$

where

$$g_0(z) = \prod_j \left(\frac{1}{z - z_j} \right)$$

The contour C is deformed to envelope all poles of $g_0(z)$



It follows that

$$S^F(\tau) = \frac{1}{\beta} \sum_{ik_n} g_0(ik_n)\, e^{ik_n\tau} = -\int_C \frac{dz}{2\pi i}\, n_F(z)\, g_0(z)\, e^{iz\tau} =$$

$$= \sum_j \operatorname*{Res}_{z=z_j} \left[g_0(z) \right] n_F(z_j)\, e^{z_j \tau} - \oint_{C_\infty} \frac{dz}{2\pi i}\, n_F(z)\, g_0(z)\, e^{z\tau}$$

The C_∞ contribution:

let $z = Re^{i\theta}$ $\quad R \to \infty \quad 0 < \tau < \beta$

$$n_F(z)e^{\tau z} = \frac{e^{\tau z}}{e^{\beta z}+1} \sim \begin{cases} e^{(\tau-\beta)\,\mathrm{Re}\, z} \xrightarrow[R\to\infty]{} 0 & \mathrm{Re}\, z > 0 \\ e^{\tau\,\mathrm{Re}\, z} \xrightarrow[R\to\infty]{} 0 & \mathrm{Re}\, z < 0 \end{cases}$$

thus $\displaystyle\oint_{C_\infty} \frac{dz}{2\pi i}\, n_F(z)\, \rho_0(z)\, e^{z\tau} = 0$

Finally,

$$\boxed{S_0^F(\tau) = \sum_j \operatorname*{Res}_{z=z_j}\left[\rho_0(z)\right] n_F(z_j)\, e^{z_j\tau}}$$

Similarly for bosons

$$\boxed{S_0^B(\tau) = -\sum_j \operatorname*{Res}_{z=z_j}\left[\varphi_0(z)\right] n_B(z_j)\, e^{z_j\tau}}$$

For noninteracting system

$$G(i\omega_n) = \frac{1}{i\omega_n - \tilde{\varepsilon}_k}$$

a simple pole at $z = \tilde{\varepsilon}_k$

$G(z) = \dfrac{1}{z - \tilde{\varepsilon}_k}$

$\tilde{\varepsilon}_k = \varepsilon_k - \mu$

Let's compute for fermions

$$\boxed{n_k} = \langle c_k^+ c_k \rangle = \lim_{\tau\to 0^-} G_k(\tau) =$$

where $G_k(\tau) = -\langle T_\tau c_k(\tau) c^+(0)\rangle = -\theta(\tau)\langle c_k(\tau) c_k^+(0)\rangle + \theta(-\tau)\langle c_k^+(0) c_k(\tau)\rangle$

$$= \lim_{\tau\to 0^-} \frac{1}{\beta}\sum_{n=-\infty}^{\infty} e^{-\frac{i\pi(2n+1)}{\beta}\tau}\, G(ik_n) =$$

$$= \lim_{\tau\to 0} \operatorname*{Res}_{z=\tilde{\varepsilon}_k}\left[\frac{1}{z-\tilde{\varepsilon}_k}\right] n_F(\tilde{\varepsilon}_k)\, e^{-\tilde{\varepsilon}_k\tau} =$$

$$= n_F(\tilde{\varepsilon}_k) = \boxed{\frac{1}{e^{\beta(\varepsilon_k-\mu)}+1}} \quad \square$$