Show that

$$\coth x - \frac{1}{x} = \sum_{m=1}^{\infty} \frac{2x}{x^2 + m^2\pi^2}$$

and

$$\frac{\sin\theta}{\theta} = \prod_{m=1}^{\infty}\left(1 - \frac{\theta^2}{m^2\pi^2}\right)$$

Theorem: $f(z)$ meromorphic function,

C – a contour surrounding zeros of $\sin \pi z$ at $z = \varrho, \varrho+1, \dots, n \in \mathbb{N}$

If poles of $f(z)$ and zeros of $\sin \pi z$ do not coincide

$$\sum_{m=\varrho}^{n} f(m) = \frac{1}{2\pi i}\oint_C \pi \cot g(\pi z) f(z)\,dz - \sum_{\substack{\text{poles in } C \\ \text{of } f(z)}} \operatorname{Res}\left[\pi \cot g(\pi z) f(z)\right]$$

proof

$$\oint_C \pi \cot g(\pi z) f(z)\, dz = 2\pi i \sum (\text{all residua}) =$$

$$= 2\pi i \left(\sum_{m=\varrho}^{n} f(m) + \sum_{\substack{\text{poles in } C \\ \text{of } f(z)}} \operatorname{Res}\left[\pi \cot g(\pi z) f(z)\right]\right)$$

$\square$

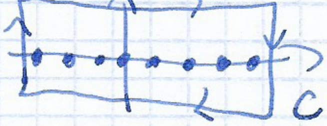
Let $$f(z) = \frac{2x}{x^2 + \pi^2 z^2}$$

Then

$$\sum_{m=-N}^{N} \frac{2x}{x^2 + m^2\pi^2} = \frac{1}{2\pi i}\oint_C \pi \cot g(\pi z) f(z)\, dz - \sum_{\text{poles of } f(z) \text{ in } C} \operatorname{Res}\left[\pi \cot g(\pi z) f(z)\right]$$

a contour enveloping points $z = -N, -N+1, \dots, N-1, N$,

Let it be rectangle of infinite length and height



C

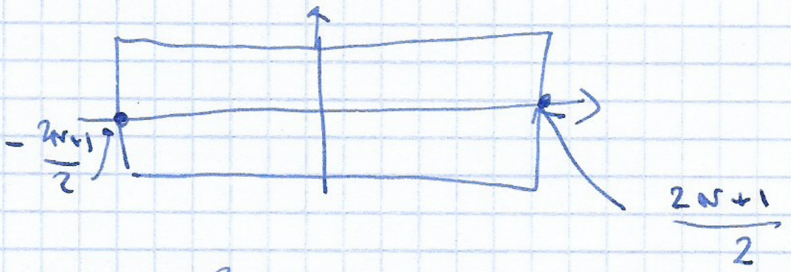
①

When the rectangle C goes to infinity

$$\left| \frac{1}{2\pi i} \oint_C \pi \cot(\pi z) f(z)\,dz \right| \leq \frac{1}{2} \oint_C |\cot(\pi z)| \left| \frac{2x}{x^2 + z^2\pi^2} \right| |dz| \underset{|z|\to\infty}{\longrightarrow} 0$$

Indeed, we note $\quad z = x + iy$

$$|\cot(\pi z)| = \frac{|\cos(\pi z)|}{|\sin(\pi z)|} = \left(\frac{\cos^2 \pi x + \sinh^2 \pi y}{\sin^2 \pi x + \sinh^2 \pi y} \right)^{1/2}$$



$N \to \infty$

then $\cos \pi\left(\frac{2N+1}{2}\right) \underset{N\to\infty}{\longrightarrow} 0$, $\sin\left(\frac{2N+1}{2}\right) \underset{N\to\infty}{\longrightarrow} 1$,

then on vertical sides of the rectangle

$$|\cot z| = \left| \frac{\sinh^2 \pi y}{1 + \sinh^2 \pi y} \right|^{1/2} = |\tanh \pi y| \leq 1$$

On horizontal sides

$$\lim_{|z|\to\infty} |\cot \pi z| = 1$$

so the integration function is $\sim \frac{1}{|z|^2}$ and vanishes at large z.

Therefore, for infinite rectangle

$$\sum_{m=-\infty}^{\infty} \frac{2x}{x^2 + m^2\pi^2} = -\operatorname{Res}\left[\frac{\pi \cot(\pi z)\, 2x}{x^2 + z^2\pi^2} \right]\Bigg|_{z = \pm \frac{ix}{\pi}} =$$

$$= -\frac{2x}{\pi} \left[\frac{\cot\left(\frac{ix}{\pi}\right)}{\frac{2ix}{\pi}} + \frac{\cot\left(-\frac{ix}{\pi}\right)}{-\frac{2ix}{\pi}} \right] = 2i \cot(ix) = 2\coth(x)$$

Thus, $\quad 2 \sum_{m=1}^{\infty} \frac{2x}{x^2 + m^2\pi^2} + \frac{2}{x} = 2\coth(x)$

and

$$\boxed{\coth(x) - \frac{1}{x} = \sum_{m=1}^{\infty} \frac{2x}{x^2 + m^2\pi^2}}$$

$\square$

To get the second result integrate the former on both sides

$$\int_0^x dx \left(\coth x - \frac{1}{x}\right) = \int_0^x dx \sum_{n=1}^{\infty} \frac{2x}{x^2 + n^2\pi^2}$$

$$\ln\left(\frac{\sinh x}{x}\right) = \sum_{m=1}^{\infty} \ln\left(1 + \frac{x^2}{m^2\pi^2}\right) =$$

$$= \ln \prod_{m=1}^{\infty} \left(1 + \frac{x^2}{m^2\pi^2}\right)$$

hence $$\boxed{\frac{\sinh x}{x} = \prod_{m=1}^{\infty} \left(1 + \frac{x^2}{m^2\pi^2}\right)}$$

Putting $x = i\theta$, $\theta \in \mathbb{R}$

$$\boxed{\frac{\sin\theta}{\theta} = \prod_{m=1}^{\infty} \left(1 - \frac{\theta^2}{m^2\pi^2}\right)}$$