SPIN SINGLET

$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B \ni |\Psi\rangle = \frac{1}{\sqrt{2}}\left[|\uparrow\rangle_A \otimes |\downarrow\rangle_B - |\downarrow\rangle_A \otimes |\uparrow\rangle_B \right]$$

$$\dim \mathcal{H} = \underbrace{\dim \mathcal{H}_A}_{=2} \cdot \underbrace{\dim \mathcal{H}_B}_{=2} = 4$$

ad. a) Measurement operator acting on A particle

$$\hat{S}_A^z \otimes \hat{1}_B = \frac{\hbar}{2} \hat{P}_A^{\uparrow} \otimes \hat{1}_B - \frac{\hbar}{2} \hat{P}_A^{\downarrow} \otimes \hat{1}_B =$$

$$= \frac{\hbar}{2} \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}_A}_{= {}_A|\uparrow\rangle\langle\uparrow|_A} \otimes \hat{1}_B - \frac{\hbar}{2} \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}_A}_{= {}_A|\downarrow\rangle\langle\downarrow|_A} \otimes \hat{1}_B$$

Probability to get $S_A^z = +\frac{\hbar}{2}$

$$P_{A\uparrow} = \langle\Psi| \hat{P}_A^{\uparrow} |\Psi\rangle =$$

$$= \frac{1}{2}\left({}_A\langle\uparrow| {}_B\langle\downarrow| - {}_A\langle\downarrow| {}_B\langle\uparrow| \right) |\uparrow\rangle_A {}_A\langle\uparrow| \otimes \hat{1}_B \left(|\uparrow\rangle_A |\downarrow\rangle_B - |\downarrow\rangle_A |\uparrow\rangle_B \right) =$$

$$= \frac{1}{2} \underbrace{{}_A\langle\uparrow|\uparrow\rangle_A}_{=1} \underbrace{{}_A\langle\uparrow|\uparrow\rangle_A}_{=1} \underbrace{{}_B\langle\downarrow|\downarrow\rangle}_{=1}$$

$$\boxed{= \frac{1}{2}}$$

ad. b) After a measurement on B subsystem it has collapsed

$$|\Psi\rangle \longrightarrow |\phi\rangle = \mathcal{N} \, \hat{1}_A \otimes \hat{P}_B^{\uparrow} |\Psi\rangle$$

$\uparrow$ normalization

$$\hat{1}_A \otimes \hat{P}_B^{\uparrow} = \hat{1}_A \otimes |\uparrow\rangle_B \langle\uparrow|_B \frac{1}{\sqrt{2}}\left[|\uparrow\rangle_A |\downarrow\rangle_B - |\downarrow\rangle_A |\uparrow\rangle_B \right] = -\frac{1}{\sqrt{2}} |\downarrow\rangle_A |\uparrow\rangle_B$$

$$1 = \langle\phi|\phi\rangle = \mathcal{N}^2 \frac{1}{2} \underbrace{{}_A\langle\downarrow| {}_B\langle\uparrow| \, |\downarrow\rangle_A |\uparrow\rangle_B}_{=1} \rightarrow \mathcal{N} = \sqrt{2} e^{i\varphi}$$

$\uparrow$ arbitrary phase factor

Measurement of $\hat{S}_A^z$ will give $\quad |\phi\rangle = -|\downarrow\rangle_A |\uparrow\rangle_B$

$$S_A^z = \langle\phi| \hat{S}_A^z \otimes \hat{1}_B |\phi\rangle = {}_A\langle\downarrow| {}_B\langle\uparrow| \left(\frac{\hbar}{2} |\uparrow\rangle_{AA}\langle\uparrow| \otimes \hat{1}_B - \frac{\hbar}{2} |\downarrow\rangle_{AA}\langle\downarrow| \otimes \hat{1}_B \right) |\downarrow\rangle_A |\uparrow\rangle_B =$$

$$= -\frac{\hbar}{2} \underbrace{{}_A\langle\downarrow|\downarrow\rangle_A}_{=1} \underbrace{{}_A\langle\downarrow|\downarrow\rangle_A}_{=1} \underbrace{{}_B\langle\uparrow|\uparrow\rangle_B}_{=1} = -\frac{\hbar}{2}$$