WHY STATISTICAL OPERATOR ?

Pure state – description by $\hat{P}_\psi = |\psi\rangle\langle\psi|$

$$i\hbar\,\partial_t|\psi\rangle = \hat{H}|\psi\rangle$$

$$\langle\hat{A}\rangle = \langle\psi|\hat{A}|\psi\rangle = \sum_i \langle\psi|\hat{A}|i\rangle\langle i|\psi\rangle = \sum_i \langle i|\psi\rangle\langle\psi|\hat{A}|i\rangle =$$

$$= \mathrm{Tr}\left[\,|\psi\rangle\langle\psi|\hat{A}\right] = \mathrm{Tr}\left[\hat{P}_\psi\hat{A}\right] = \mathrm{Tr}\left[\hat{A}\hat{P}_\psi\right]$$

projector $\hat{P}_\psi^2 = \hat{P}_\psi, \quad \hat{P}_\psi^\dagger = \hat{P}_\psi$

$$i\hbar\,\partial_t\hat{P}_\psi = [\hat{H}, \hat{P}_\psi]$$ – Schrödinger picture ✓

Mixed state – description by $\hat{\rho} = \sum_\alpha p_\alpha \hat{P}_\alpha$ – convex combination of projectors

$\langle\hat{A}\rangle = \mathrm{Tr}[\hat{A}\hat{\rho}], \quad i\hbar\,\partial_t\hat{\rho} = [\hat{H}, \hat{\rho}]$ – Schrödinger picture

$\hat{\rho} = \sum_{nm} \rho_{nm}|n\rangle\langle m| \qquad \hat{H}|n\rangle = i\hbar\,\partial_t|n\rangle$

proof: $\qquad\qquad\qquad\qquad \langle n|\hat{H} = -i\hbar\,\partial_t\langle n|$

$$\frac{d\hat{\rho}}{dt} = \sum_{nm}\rho_{nm}\left(\partial_t|n\rangle\langle m| + |n\rangle\partial_t\langle m|\right) = -\frac{i}{\hbar}[\hat{H}, \hat{\rho}] \quad \square$$

Mixed state $\Longleftrightarrow$ a state where each wave function ψ_α appears with probability p_α.

Why the wave function superposition does not work:

Let $|\psi\rangle = \sum_a e^{i\varphi_a}\sqrt{p_a}\,|\psi_a\rangle$, φ_a – unknown phase factors

This is meaningless.

We can try to average over φ_α $\left(\frac{1}{2\pi}\int_0^{2\pi} d\varphi_\alpha\, e^{i\varphi_\alpha} = 0\right)$
but result vanishes.

By taking $|\psi_\alpha\rangle\langle\psi_\alpha|$ the $\overbrace{\text{phase factor}}^{\text{unknown}}$ is gone!

$$\hat{\rho} = \sum_\alpha p_\alpha |\psi_\alpha\rangle\langle\psi_\alpha| = \sum_{nm}\rho_{nm}|n\rangle\langle m|$$