

## (Anti)symmetrization operator as a projector

$$\hat{P}_{\{\frac{A}{F}\}} \Psi(\vec{r}_1, \dots, \vec{r}_N) = \frac{1}{N!} \sum_{\mathcal{P}} \left\{ \begin{matrix} \mathcal{P} \\ \mathcal{P} \end{matrix} \right\} \Psi(\vec{r}_{\mathcal{P}1}, \dots, \vec{r}_{\mathcal{P}N})$$

↑ sum of all permutations of a set  $\{1, \dots, N\}$

$\left\{ \begin{matrix} \mathcal{P} \\ \mathcal{P} \end{matrix} \right\} = \begin{cases} -1 & \text{fermions} \\ +1 & \text{bosons} \end{cases}$

$$\hat{P}_{\{\frac{A}{F}\}}^2 \Psi(\vec{r}_1, \dots, \vec{r}_N) = \frac{1}{N!} \frac{1}{N!} \sum_{\mathcal{P}} \sum_{\mathcal{P}'} \left\{ \begin{matrix} \mathcal{P} \\ \mathcal{P} \end{matrix} \right\} \left\{ \begin{matrix} \mathcal{P}' \\ \mathcal{P}' \end{matrix} \right\} \Psi(\vec{r}_{\mathcal{P}'\mathcal{P}1}, \dots, \vec{r}_{\mathcal{P}'\mathcal{P}N}) =$$

$\mathcal{P}'\mathcal{P}$  - group composition of  $\mathcal{P}'$  and  $\mathcal{P}$ . Since  $\left\{ \begin{matrix} \mathcal{P} \\ \mathcal{P} \end{matrix} \right\} \left\{ \begin{matrix} \mathcal{P}' \\ \mathcal{P}' \end{matrix} \right\} = \left\{ \begin{matrix} \mathcal{P}'\mathcal{P} \\ \mathcal{P}'\mathcal{P} \end{matrix} \right\}$

$$\text{sum} \rightarrow \sum_{\mathcal{P}} \sum_{\mathcal{P}'} \rightarrow \sum_{\mathcal{P}} \sum_{\mathcal{Q}}$$

$$= \frac{1}{N!} \sum_{\mathcal{P}} \left( \frac{1}{N!} \sum_{\mathcal{Q}} \left\{ \begin{matrix} \mathcal{Q} \\ \mathcal{Q} \end{matrix} \right\} \Psi(\vec{r}_{\mathcal{Q}1}, \dots, \vec{r}_{\mathcal{Q}N}) \right) = \hat{P}_{\{\frac{A}{F}\}} \Psi(\vec{r}_1, \dots, \vec{r}_N)$$

□

### Example

$$\hat{J}_{\{R\}} \psi(\vec{r}_1, \vec{r}_2) = \frac{1}{2!} (\psi(\vec{r}_1, \vec{r}_2) \pm \psi(\vec{r}_2, \vec{r}_1))$$

$$\begin{aligned} \hat{J}_{\{R\}}^2 &= \frac{1}{2!} \left( \hat{J}_{\{R\}} \psi(\vec{r}_1, \vec{r}_2) \pm \hat{J}_{\{R\}} \psi(\vec{r}_2, \vec{r}_1) \right) = \\ &= \frac{1}{2!} \left( \frac{1}{2!} (\psi(\vec{r}_1, \vec{r}_2) \pm \psi(\vec{r}_2, \vec{r}_1)) \pm \frac{1}{2!} (\psi(\vec{r}_2, \vec{r}_1) \pm \psi(\vec{r}_1, \vec{r}_2)) \right) = \\ &= \frac{1}{4} \left( 2 \psi(\vec{r}_1, \vec{r}_2) \pm 2 \psi(\vec{r}_2, \vec{r}_1) \right) = \\ &= \frac{1}{2} (\psi(\vec{r}_1, \vec{r}_2) \pm \psi(\vec{r}_2, \vec{r}_1)) = \hat{J}_{\{R\}} \psi(\vec{r}_1, \vec{r}_2) \end{aligned}$$