(Anti)Symmetrization operator as a projector

$$\hat{S}_{\{B \atop F\}} \Psi(\vec{r}_1 .. \vec{r}_N) = \frac{1}{N!} \sum_{P} \xi^{P} \Psi(\vec{r}_{P_1} ... \vec{r}_{P_N})$$

$\sum_P$ ↑ sum of all permutation of a set $\{1, .., N\}$

$$\xi = \begin{cases} -1 & \text{fermions} \\ +1 & \text{bosons} \end{cases}$$

$$\hat{S}^2_{\{B \atop F\}} \Psi(\vec{r}_1 .. \vec{r}_N) = \frac{1}{N!} \frac{1}{N!} \sum_{P} \sum_{P'} \xi^{P} \xi^{P'} \Psi(\vec{r}_{P'P_1} .. \vec{r}_{P'P_N}) =$$

$P'P$ - group composition of P' and P. Since $\xi^{P} \xi^{P'} = \xi^{P+P'}$

Sum $\sum_{P} \sum_{P'} \to \sum_{P} \sum_{Q}$

$$= \frac{1}{N!} \underbrace{\sum_{P}}_{N!} \underbrace{\left(\frac{1}{N!} \sum_{Q} \xi^{Q} \Psi(\vec{r}_{Q_1} .. \vec{r}_{Q_N}) \right)}_{\hat{S}_{\{B \atop F\}} \Psi(\vec{r}_1 .. \vec{r}_N)} = \hat{S}_{\{B \atop F\}} \Psi(\vec{r}_1 .. \vec{r}_N)$$

$\square$

Example

$$\hat{S}_{\left\{ B \atop F \right\}} \psi(\vec{r}_1, \vec{r}_2) = \frac{1}{2!} \left(\psi(\vec{r}_1 \vec{r}_2) \pm \psi(\vec{r}_2 \vec{r}_1) \right)$$

$$\hat{S}_{\left\{ B \atop F \right\}}^2 = \frac{1}{2!} \left(\hat{S}_{\left\{ B \atop F \right\}} \psi(\vec{r}_1, \vec{r}_2) \pm \hat{S}_{\left\{ B \atop F \right\}} \psi(\vec{r}_2, \vec{r}_1) \right) =$$

$$= \frac{1}{2!} \left(\frac{1}{2!} \left(\psi(\vec{r}_1 \vec{r}_2) \pm \psi(\vec{r}_2 \vec{r}_1) \right) \pm \frac{1}{2!} \left(\psi(\vec{r}_2 \vec{r}_1) \pm \psi(\vec{r}_1 \vec{r}_2) \right) =$$

$$= \frac{1}{4} \left\{ 2 \psi(\vec{r}_1, \vec{r}_2) \pm 2 \psi(\vec{r}_2 \vec{r}_1) \right\} =$$

$$= \frac{1}{2} \left(\psi(\vec{r}_1, \vec{r}_2) \pm \psi(\vec{r}_2 \vec{r}_1) \right) = \hat{S}_{\left\{ B \atop F \right\}} \psi(\vec{r}_1 \vec{r}_2)$$