Tensor product

A, B - vector spaces $|a_i\rangle \in A$, $|b_i\rangle \in B$

Tensor product of $A \otimes B$ is a set of possible pairs $|a\rangle \otimes |b\rangle$ and their linear combinations, such that:

1) $(|a_1\rangle + |a_2\rangle) \otimes |b\rangle = |a_1\rangle \otimes |b\rangle + |a_2\rangle \otimes |b\rangle$
2) $|a\rangle \otimes (|b_1\rangle + |b_2\rangle) = |a\rangle \otimes |b_1\rangle + |a\rangle \otimes |b_2\rangle$
3) $(\lambda |a\rangle) \otimes |b\rangle = |a\rangle \otimes (\lambda |b\rangle) = \lambda (|a\rangle \otimes |b\rangle)$

If $\{|\alpha_i\rangle\}$ base of A and $\{|\beta_i\rangle\}$ base of B then $\{|\alpha_k\rangle \otimes |\beta_l\rangle\}$ base of $A \otimes B$.

$\mathrm{Dim}(A \otimes B) = (\mathrm{Dim}\, A) \cdot (\mathrm{Dim}\, B)$

Scalar product

$$[|\Phi\rangle \otimes |\Psi\rangle] \cdot [|\varphi\rangle \otimes |\psi\rangle] = \langle \Phi | \varphi \rangle \cdot \langle \Psi | \psi \rangle$$

$|\Phi\rangle, |\varphi\rangle \in A$, $|\Psi\rangle, |\psi\rangle \in B$.

M. Reed, B. Simon p.51 (32)

$L^2(\mathbb{R}^3 \times \mathbb{R}^3)$ contains functions that are not in the Cartesian product $L^2(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$. One can show that $L^2(\mathbb{R}^3 \times \mathbb{R}^3) \underset{\text{isomorphic}}{\simeq} L^2(\mathbb{R}^3) \otimes L^2(\mathbb{R}^3)$

This the reason why we take in general $\mathcal{H}_1 \otimes \mathcal{H}_2$ for two subsystems

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$|v\rangle = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ $|w\rangle = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$

$$|v\rangle \otimes |w\rangle = \begin{pmatrix} v_1 w_1 \\ v_1 w_2 \\ v_1 w_3 \\ \hline v_2 w_1 \\ v_2 w_2 \\ v_2 w_3 \end{pmatrix} \quad 2 \times 3 = 6$$

$$\hat{A} \otimes \hat{B} = \left(\begin{array}{ccc|c} a_{11} b_{11} & a_{11} b_{12} & a_{11} b_{13} & a_{12} b_{11} \; \dots \\ a_{11} b_{21} & a_{11} b_{22} & a_{11} b_{23} & a_{12} b_{21} \\ a_{11} b_{31} & a_{11} b_{32} & a_{11} b_{33} & a_{12} b_{31} \\ \hline a_{21} b_{11} & a_{21} b_{12} & a_{21} b_{13} & a_{22} b_{11} \\ a_{21} b_{21} & a_{21} b_{22} & a_{21} b_{23} & a_{22} b_{21} \\ a_{21} b_{31} & a_{22} b_{32} & a_{21} b_{33} & a_{22} b_{31} \end{array}\right)$$

$(2 \times 3) \times (2 \times 3) = 6 \times 6$

$(\hat{A} \otimes \hat{B})(|v\rangle \otimes |w\rangle) = (\hat{A}|v\rangle) \otimes (\hat{B}|w\rangle)$

$$|v\rangle \oplus |w\rangle = \begin{pmatrix} v_1 \\ v_2 \\ \hline w_1 \\ w_2 \\ w_3 \end{pmatrix} \quad 2 + 3 = 5$$

$$\hat{A} \oplus \hat{B} = \left(\begin{array}{cc|ccc} a_{11} & a_{12} & & 0 & \\ a_{21} & a_{22} & & & \\ \hline & & b_{11} & b_{12} & b_{13} \\ & 0 & b_{21} & b_{22} & b_{23} \\ & & b_{31} & b_{32} & b_{33} \end{array}\right)$$

$(2+3) \times (2+3) = 5 \times 5$

$(\hat{A} \oplus \hat{B})(|v\rangle \oplus |w\rangle) = (\hat{A}|v\rangle) \oplus (\hat{B}|w\rangle)$