Topics in Many Body Theory

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Rules

- Lectures and Tutorials are in Tuesdays at 11:15-13:00 in 1.02 room and in Thursdays at 8:15-10:00 in 1.01 room.
- Way of passing the course: There is an oral exam based on the theory from lectures (listed in this file) and problems solved on tutorials and homeworks (listed in this file). You are allowed to posses your handwritten notes from lectures and with solved problems during your exam and supposed to present 2-3 given topics in concise and short forms.

1 Week I, 27/02-05/03/2023

1.1 Lecture PJ

Review of the quantum many body formalism, notion or bosonic and fermionic coherent states, general expression for bosonic coherent states.

1.2 Tutorial PJ

1. Multidimensional Gaussian integrals on $\mathbb{R}^n$, moments of the Gaussian distribution, Wick's theorem.

2 Week II, 06-12/03/2023

2.1 Lecture PJ

Bosonic coherent states ctd, Grassmann variables (basic notions and properties, conjugation, differentiation, integration), fermonic coherent states.

2.2 Tutorial PJ

1. Perturbed Gaussian measure, perturbative expansions, Feynman diagrams for the normalization factor and the 2-point functions.

2.3 Homework problems

1. Demonstrate the closure relation $\int \prod_\alpha \frac{d\phi^*_\alpha d\phi_\alpha}{2\pi i} e^{-\sum_{\alpha'} \phi^*_{\alpha'}\phi_{\alpha'}} |\phi\rangle\langle\phi|$ for bosonic coherent states.
2. Consider $\langle \hat{N}\rangle = \frac{\langle\phi|\hat{N}|\phi\rangle}{\langle\phi|\phi\rangle}$, where $|\phi\rangle$ is a bosonic coherent state, and $\hat{N}$ denotes the total particle number operator. Express $\sigma^2 = \langle \hat{N}^2\rangle - \langle\hat{N}\rangle^2$ by the eigenvalues $\{\phi_\alpha\}$ corresponding to $|\phi\rangle$.
3. (Not obligatory) Consider the perturbed Gaussian measure $\frac{1}{Z(\lambda)} e^{-A(\vec{x},\lambda)}$. Notation follows the one used in the class and $V(\vec{x}) = \frac{1}{4!}\sum_{i=1}^N x_i^4$. Draw the Feynman graphs corresponding to the two-point function $\langle x_{i_1 i_2}\rangle_\lambda$ up to order λ^2. Use the given fact (linked cluster theorem) that the non-connected contributions cancel. Write down the analytical expressions corresponding to the diagrams and try to figure out the corresponding coefficients from combinatorial arguments.

3 Week III, 13-19/03/2023

3.1 Lecture PJ

Fermionic coherent states ctd: closure relation, trace formula. Coherent states for fermions and bosons summary. Propagator in QM (introduction).

3.2 Tutorial KB

1. *Reminder of Grassmann algebra* - generators, anticommutation rules, reflection authomorphism P, even and odd Grassmann subspaces.
2. *Formal definitions* of conjugation, differentiation and integration in Grassmann algebra.
3. *Grassmann calculus* Let $f(\xi) = f_0 + f_1\xi$ and $A(\xi^*, \xi) = a_0 + a_1\xi + \bar{a}_1\xi^2 + a_{12}\xi^*\xi$. Compute
   (a) $\partial A(\xi^*,\xi)/\partial\xi$
   (b) $\partial A(\xi^*,\xi)/\partial\xi^*$
   (c) $\partial^2 A(\xi^*,\xi)/\partial\xi^*\partial\xi$
   (d) $\int d\xi f(\xi)$
   (e) $\int d\xi A(\xi^*,\xi)$
   (f) $\int d\xi^* A(\xi^*,\xi)$
   (g) $\int d\xi^* d\xi A(\xi^*,\xi)$
4. *Grassmann delta Dirac distribution* Show that $\delta(\xi,\xi) = \int \eta \exp(-\eta(\xi - \xi')$ represents Dirac delta function.
5. *Grassmann scalar product* Check that $\langle f|g\rangle = \int d\xi d\xi^* \exp(-\xi\xi^*) f^*(\xi^*) g(\xi)$ represents a natural scalar product between functions f and g.
6. Grassmann-Gaussian one-variable integral Show that $\int d\xi^* d\xi \exp(-\xi^* a \xi) = a$. Compare this result with Gaussian integral over complex numbers.

7. Change of variable From $\int \eta\eta = 1$ and by changing the variable $\eta = \alpha\xi$ show for a one Grassmann variable that the Jacobian $d(\alpha\xi) = (d\xi)/\alpha$. Apply this result to compute the Gaussian integral $\int d\xi^* d\xi \exp(-\xi^* a\xi) = a$ again.

8. Show that

$$\int \prod_{i=1}^{n} d\eta_i^* d\eta_i e^{-\sum_{ij} \eta_i^* H_{ij}\eta_j + \sum_i \xi_i^* \eta_i + \xi_i \eta_i^*} =$$

$$= [\mathrm{Det}H] e^{\sum_{ij} \xi_i^* (H^{-1})_{ij} \xi_j}.$$

3.3 Homework problems

1. Show that

$$\int \prod_{i=1}^{n} d\eta_i^* d\eta_i \eta_k \eta_l^* e^{-\sum_{ij} \eta_i^* H_{ij} \eta_j i} =$$

$$= [\mathrm{Det}H](H^{-1})_{kl}.$$

2. Review the demonstration of the closure relation using fermionic coherent states: $1 = \int \prod_\alpha d\xi_\alpha^* d\xi_\alpha e^{-\sum_\alpha \xi_\alpha^* \xi_\alpha} |\xi\rangle\langle\xi|$. Evaluate the final integral left out in the proof given in the lecture.

4 Week IV, 20-26/03/2023

4.1 Lecture PJ

Partition function and propagator in imaginary time for a single quantum particle; the classical limit. Coherent state path integral for the partition function of many-body systems.

4.2 Tutorial KB

1. *Feynman path integral in quantum mechanics* - Express a propagator in one-particle quantum mechanics in terms of a path integral with a classical action

$$G(x,t;x',t') = \int \mathcal{D}[x(t)] e^{\frac{i}{\hbar}\int_t^{t'} d\tau (\frac{1}{2}m\dot{x}(t)^2 - V(x))}$$

$$= \int \mathcal{D}[x(t)] e^{\frac{i}{\hbar} S[x(\tau)]}.$$

Discuss mathematical meaning of this functional integration.

2. *Green function (propagator) for free particle* - Derive an expression for the Green function (propagator) of one-particle in one dimension with $V(x) = 0$. Do it in two ways, directly from the definition in quantum mechanics and from the path integration in discrete version.

4.3 Homework problems

1. *Normal ordering* - Consider a single quantum particle governed by the hamiltonian $H = \frac{\vec{p}^2}{2m} + V(\vec{x})$. Write down the complete (double series) expression for : $e^{-i\frac{H}{\hbar}\epsilon}$:. Compare it with the corresponding expression for $e^{-i\frac{H}{\hbar}\epsilon}$. Compare in particular the two expressions, when truncated at order ϵ and at order ϵ^2.

2. *Quamtum mechanical operators and path integrals* - For a given trajectory we can define a classical momentum $m\dot{x}(t)$. Let's define at t_f a quantum mechanical momentum operator in one dimension as

$$\hat{p}\Psi(x_f, t_f) =$$

$$= \int_{-\infty}^{\infty} dy \int \mathcal{D}[x(t)] m\dot{x}(t_f) e^{\frac{i}{\hbar} S[x(\tau)]} \Psi(y, t_i).$$

Show that $\hat{p} = -i\hbar d/dx$.

5 Week V, 27/03-02/04/2023

5.1 Lecture PJ

Evaluation of the coherent path integral for the partition function of noninteracting Bose and Fermi systems. Linear response theory and the Kubo formula (review). N-body real-time Green's functions, thermal (imaginary time) Green's functions and their path integral representation.

5.2 Tutorial KB

1. *Statistical mechanics and path integrals* - Formulation of the partition function in canonical ensemble in terms of the path integral

$$Z = \mathrm{Tr} e^{-\beta\hat{H}} =$$

$$= \int_{x(0)=x(\beta)} \mathcal{D}[x(\tau)] e^{-\frac{1}{\hbar} S[x(\tau)]},$$

where

$$S[x(\tau)] = \int_0^\beta d\tau \left(\frac{1}{2m} \left(\frac{dx(\tau)}{d\tau} \right)^2 + V(x(\tau)) \right).$$

2. *Partition function for a harmonic oscillator* - Derive the partition function for a harmonic oscillator in quantum mechanics, i.e. $V(x) = \frac{1}{2}m\omega^2 x^2$. To compute the path integral use the Fourier series expansion in Matsubara frequencies $\omega_n = 2n\pi/\beta$ and the integration measure constant find from the free particle problem.

3. *2-point correlation functions for harmonic oscillator* - Two point correlation function is defined

$$G(\tau) = \frac{1}{Z} \mathrm{Tr} \left[e^{-\beta\hat{H}} \hat{x}(\tau)\hat{x}(0) \right],$$

where

$$\hat{x}(\tau) = e^{\frac{\hat{H}\tau}{\hbar}} \hat{x} e^{-\frac{\hat{H}\tau}{\hbar}},$$

with $0 \leqslant \tau < \beta\hbar$. The path integral representation of this correlation function is

$$G(\tau) = \frac{1}{Z}\int_{x(0)=x(\beta)} \mathcal{D}[x(\tau)]\; x(\tau)x(0)\; e^{-\frac{1}{\hbar}S[x(\tau)]}.$$

Determine this functional integral using the Fourier series expansion in Matsubara frequencies.

5.3 Homework problems

1. By direct calculation find the inverse of the matrix $S^{(\alpha)}$ discussed in the lecture and yielding the expression for the thermal Green's function of the non-interacting Fermi or Bose system.
   Hint: It may be helpful to perform the calculation for a small M (e.g. $M = 4$) first.

2. Show by direct means based on standard quantum statistical mechanics that the 2-point correlation function in harmonic oscillator takes the form

$$G(\tau) = \frac{\hbar}{2m\omega}\frac{\cosh\left[\left(\frac{\beta\hbar}{2}-\tau\right)\omega\right)\right]}{\sinh\left[\frac{\beta\hbar\omega}{2}\right]}.$$

   Evaluate in details all steps needed in the problem 3 of the last tutorial.

6 Week VI, 03-09/04/2023

6.1 Lecture

No lecture in this Easter week.

6.2 Tutorial KB

1. *Evaluation of Matsubara sums* - Derive the contour integral equations for fermionic (F) and bosonic (B) Matsubara sums:

$$S^F(\tau) = \frac{1}{\beta}\sum_{ik_n} g(ik_n)e^{ik_n\tau} = -\oint_C \frac{dz}{2\pi i} n_F(z)g(z)e^{z\tau},$$

$$S^B(\tau) = \frac{1}{\beta}\sum_{i\omega_n} g(i\omega_n)e^{i\omega_n\tau} = \oint_C \frac{dz}{2\pi i} n_B(z)g(z)e^{z\tau},$$

   where $k_n = (2n+1)\pi/\beta$, $\omega_n = 2n\pi/\beta$, and $n_{F(B)}(z) = 1/(e^{\beta z} \pm 1)$.

2. *Matsubara sum over functions with singe poles* - Derive a fermionic Matsubara sum for a function with single poles

$$g_0(z) = \prod_j \left(\frac{1}{z-z_j}\right).$$

   Apply this result to the fermionic non-interacting Green function

$$G^0_{\mathbf{k}}(ik_n) = \frac{1}{ik_n - (\epsilon_{\mathbf{k}} - \mu)},$$

   and derive probability of occupation of a single state $\mathbf{k}$.

3. *Matsubara sum over functions with a branch cut* - Derive a fermionic Matsubara sum for a function with a branch cut along the real axis $g(\epsilon)$. Derive probability of occupation of a single state $\mathbf{k}$.

6.3 Homework problems

1. *Matsubara sum over functions with singe poles* - Derive a bosonic Matsubara sum for a function with single poles

$$g_0(z) = \prod_j \left(\frac{1}{z-z_j}\right).$$

   Apply this result to the bosonic non-interacting Green function

$$G^0_{\mathbf{k}}(i\omega_n) = \frac{1}{i\omega_n - (\epsilon_{\mathbf{k}} - \mu)},$$

   and derive probability of occupation of a single state $\mathbf{k}$.

2. *Matsubara sum over functions with a branch cut* - Derive a bosonic Matsubara sum for a function with a branch cut along the real axis $g(\epsilon)$. derive probability of occupation of a single state $\mathbf{k}$.

3. *Other application* - Derive that

$$\operatorname{ctgh} x - \frac{1}{x} = \sum_{m=1}^{\infty} \frac{2x}{x^2 + m^2\pi^2}.$$

   Then, show that

$$\frac{\sinh x}{x} = \prod_{m=1}^{\infty}\left(1 + \frac{x^2}{m^2\pi^2}\right)$$

   and

$$\frac{\sin\theta}{\theta} = \prod_{m=1}^{\infty}\left(1 - \frac{\theta^2}{m^2\pi^2}\right).$$

   Hint: F. W. Byron and R. W. Fuller, *Matematyka w fizyce klasycznej i kwantowej*. Tom 2

7 Week VII, 10-16/04/2023

7.1 Lecture PJ

Green's functions for non-interacting fermions and Bosons, Wick's theorem, Feynman diagrams (to be continued).

7.2 Tutorial

No tutorial in this Easter week.

8 Week VIII, 17-23/04/2023

8.1 Tutorial KB

1. *Hubbard model in the atomic limit* - Find the partition function of the Hubbard model in the atomic limit

$$\hat{H} = \sum_\sigma \hat{n}_\sigma + U\hat{n}_\uparrow\hat{n}_\downarrow,$$

a) directly in the occupation number base, b) by Feynman path integrals and the Hubbard-Stratonovic transformation.

2. *Polaron model in the atomic limit* - Find the partition function of the polaron model in the atomic limit

$$\hat{H} = g \sum_\sigma (\hat{b}^\dagger + \hat{b})\hat{n}_\sigma - \sum_\sigma \mu_\sigma \hat{n}_\sigma + w\hat{b}^\dagger \hat{b}$$

by computing the Feynman path integral.

9 Week IX, 24-30/04/2023

9.1 Lecture KB

Hubbard model

&1. *Wannier functions* - Bloch theorem, Brillouin zone, Wannier functions and their properties.
&2. *Hubbard model and its derivation* - Many body Hamiltonian in terms of the second-quantized field operators for fermions, expansion of the field operators in the Wannier one-particle base, creation and annihilation operators at the lattice sites, one-body part of the Hamiltonian, hopping amplitude, two-body part of the Hamiltonian, local diagonal interaction, the Hubbard model

$$\hat{H} = \sum_{ij\sigma} t_{ij} \hat{a}^\dagger_{i\sigma} \hat{a}_{j\sigma} + \sum_i \hat{n}_{i\uparrow} \hat{n}_{i\downarrow}.$$

&3. *Extended Hubbard model* - Two-site interaction, direct Coulomb term, exchange term, and singlet hoping term.
&4 *Symmetries of the Hubbard model* - global $U(1)$ gauge symmetry, conservation of charge, spin rotation $SU(2)$ symmetry, particle-hole symmetry on bipartite lattices.

10 Week X, 01-07/05/2023

10.1 Tutorial PJ

Perturbation theory for the partition function and the grand canonical potential of interacting Fermi and Bose systems, Feynman diagrams, linked cluster theorem - proof by replica method.

10.2 Homework problems

1. A system of quantum particles is described by the (grand-canonical) hamiltonian $H = H_0 + V$, where $H_0 = \sum_\alpha (\epsilon_\alpha - \mu) a^\dagger_\alpha a_\alpha$, and V represents a two-body interaction. Consider perturbative expansion for the partition function Z up to second order in V.
a) How many terms (contractions) are there?
b) Draw all the distinct Feynman diagrams and try to figure out the corresponding numerical coefficients. Check consistency by comparing with point a). [*Hint*: There are 8 distinct diagrams of order 2.]
c) Which diagrams obtained in point b) contribute to the grand canonical potential Ω?

2. *Particle-hole symmetry* - Show that for the Hubbard model on a bipartite lattice (particle-hole transformation) $n(\mu, T) = 2 - n(U - \mu, T)$. Show that at half-filing particle-hole symmetry requires that $\mu = U/2$.

11 Week XI, 08-14/05/2023

11.1 Lecture KB

&5 *Exactly solvable limits in arbitrary dimensions* - free electron and atomic limits, *free electron limit*: discrete Fourier transform, diagonalization of the Hamiltonian, dispersion relation, examples of dispersion relations, thermodynamics of free electron on lattices, grand partition function, grand thermodynamical potential, averages, limits of a band insulator and metal, Fermi energy and the band filling, one-particle Green's function for free electrons, spectral function, density of states, physical interpretations.

11.2 Tutorial PJ

Perturbation theory and Feynman diagrams ctd - frequency/momentum representation, expansion for the Green's function.

11.3 Homework problems

1. A system of quantum particles is described by the (grand-canonical) hamiltonian $H = H_0 + V$, where $H_0 = \sum_\alpha (\epsilon_\alpha - \mu) a^\dagger_\alpha a_\alpha$, and V represents a two-body interaction. Consider perturbative expansion for the thermal Green's function G up to first order in V.
a) Write down the complete expression for the terms representing $\langle \psi_{\alpha_1}(\tau_1) \psi^*_{\alpha_2}(\tau_2) e^{-S_{int}} \rangle_0$ and draw the corresponding diagrams.
b) Show directly (at first order in V) the cancellation of the contributions represented by non-compact diagrams in the expansion of G.

2. *Dispersion relation on a square lattice* - Find the dispersion relation for free fermions on a square lattice with nearest and next nearest neighbor hopping amplitudes.

3. *Dispersion relation on a hexagonal lattice* - Find the dispersion relation for free fermions on a hexagonal lattice with nearest neighbor hopping amplitudes. Take into account that the elementary cell is made of two atoms, so you should find two symmetric dispersions in the Brillouin zone.

12 Week XII, 15-21/05/2023

12.1 Lecture KB

atomic limit: partition function in the atomic $t_{ij} = 0$ limit of the Hubbard model, grand canonical potential, single-site occupation function, non-Fermi-Dirac distribution function, $U \to 0$ and $U \to \infty$ limits, equation of motion for the retarded Green functions, exact form of the Green function and spectral function, physical interpratation.

&6. *Two site Hubbard model* - The Hilbert space of the two site Hubbard model, dimension and basis vectors for different numer of particles, ...

12.2 Tutorial PJ

Self-energy and its diagrammatic expansion, Dyson's equation, Lehmann representation of the Green's function.

12.3 Homework problems

1. *Hubbard model in atomic limit* - Find equations of motion for one and two particle Matsubara Green's functions and solve hem in the Matsubara frequency representation.

13 Week XIII, 22-28/05/2023

13.1 Lecture KB

...matrix elements and the matrix Hamiltonian for two electrons, singlet and triplet states, block diagonalization, discussion of the exact solution, exact eigenstates and eigenvalues, different limits.

&7. *t-J model* - Projection on single and double occupied sites, $\Pi_1 = \prod_i (1 - n_{i\uparrow} n_{i\downarrow})$ and $\Pi_2 = \prod_i n_{i\uparrow} n_{i\downarrow}$, canonical transformation and removing transition between single and double occupied sectors, ...

13.2 Tutorial PJ

Electronic gas (jellium model) and its ground state energy in perturbation theory.

13.3 Homework problems

1. Calculate the integrals leading to the first-order correction to the ground state energy of the jellium model, in particular the volume V_α of the overlap of two balls in 3 dimensions. Evaluation of these integrals was left out during the class.

2. *Two site Hubbard model* - Compute all matrix elements for $N = 2$ particles in the two-site Hubbard model.

14 Week XIV, 29/05-04/06/2023

14.1 Lecture KB

... explicit for of the effective t-J H hamiltonian and discussion, RVB theory and real space pairing.

&8. *Exact theorems on the Hubbard model* - Lieb theorem (1989), Koma, Tasaki theorem (1992), Nagaoka theorem (1965).

14.2 Tutorial PJ

Electronic gas (jellium model), RPA resummation for the self-energy.

14.3 Homework problems

1. Calculate the integrals leading to the first-order correction to the ground state energy of the jellium model, in particular the volume V_α of the overlap of two balls in 3 dimensions. Evaluation of these integrals was left out during the class.

2. *RVB and real-space pairing* - Consider t-J Hamiltonian with three site term

$$H_{\mathrm{t-J}} = \sum_{ij}{}' t_{ij} b^\dagger_{i\sigma} b_{j\sigma} + \sum_{ij}{}' \frac{2t_{ij}}{U} (\vec{S}_i \cdot \vec{S}_j - \frac{1}{4}\nu_i \nu_j) -$$

$$- \sum_{ijk\sigma}{}'' \frac{t_{ij} t_{jk}}{U} (b^\dagger_{i\sigma} \nu_{j\bar{\sigma}} b_{k\sigma} + b^\dagger_{i\sigma} S^{\bar{\sigma}}_j b_{k\bar{\sigma}}),$$

where $b_{i\sigma} = a_{i\sigma}(1 - n_{i\bar{\sigma}})$, etc., are projected fermionic operators. Introduce the Cooper pair operator in real-space

$$B^\dagger_{ij} = \frac{1}{\sqrt{2}} (b^\dagger_{i\uparrow} b^\dagger_{j\downarrow} - b^\dagger_{i\downarrow} b^\dagger_{j\uparrow}),$$

etc. Show that

$$H_{\mathrm{t-J}} = \sum_{ij}{}' t_{ij} b^\dagger_{i\sigma} b_{j\sigma} - \sum_{ijk\sigma}{}'' \frac{t_{ij} t_{jk}}{U} B^\dagger_{ij} B_{kj}.$$

15 Week XV, 05-11/06/2023

15.1 Lecture KB

&9. *Exact solution of the Hubbard model in infinite dimension - Dynamical Mean-Field Theory* - Brief history of mean-field approximations, Baym's criteria on the reliable approximate theory, Weiss mean-field theory for Ising model and its exactly solvable limit with infinite coordination number, different lattices and coordination numbers, construction of the exactly solvable Hubbard model in infinite coordination number, dimension, quantum rescaling of the hopping amplitude, diagramatic simplification in infinite coordination number limit, local, diagonal self-energy, cavity construction and integration out the rest of the lattice, ...

15.2 Tutorial PJ

no tutorial, holiday.

15.3 Homework problems

1. *Density of states in infinite dimensions* - On the hypercubic lattice in d-dimensions consider a nearest neighbor hopping Hamiltonian and the dispersion relation $\epsilon_{\mathbf{k}} = 2t\sum_{i=1}^{d}\cos(k_i)$. Show that the density of states is
$$\rho_d(\omega) = \sum_{\mathbf{k}} \delta(\omega - \epsilon_{\mathbf{k}}) =$$
$$= \int_{-\infty}^{\infty} \frac{du}{2\pi} \prod_{i=1}^{d} \int_{-\pi}^{\pi} \frac{dk_i}{2\pi} e^{iu(\omega + 2t\sum_{i=1}^{d}\cos(k_i))} =$$
$$= \int_{-\infty}^{\infty} \frac{du}{2\pi} e^{iu\omega} [J_0(2ut)]^d,$$
where $J_0(x)$ is the Bessel function. Make a hopping rescaling $t \to t^*/\sqrt{2d}$, expand the integrand, and find the density of states in the $d \to \infty$ limit. Ans. $\rho_{d=\infty}(\omega) = (1/\sqrt{2\pi}t^*)e^{-\omega^2/2t^{*2}}$.

16 Week XVI, 12-18/06/2023

16.1 Lecture KB

..., terms which are finite in the infinite dimension, linked cluster theorem and resummation of the cavity partition function, Dyson equation in k-space and in real-space, self-consistency condition, different methods for solving DMFT equations.

16.2 Tutorial PJ

Electronic gas (jellium model) and its ground state energy in RPA, the pair bubble and the Thomas-Fermi screening length.

17 Exam questions - PJ + KB

1. Coherent state functional integrals.
2. Wick theorem.
3. Perturbation theory for the partition function, linked cluster theorem.
4. Perturbation theory for the imaginary time Green's function.
5. Ground state energy of the electron gas. Random phase approximation for the self-energy and the ground state energy.
6. Derive the Hubbard model from the many-body Schroedinger Hamiltonian. Discuss terms which are taken into account and which are neglected.
7. Discuss $U(1)$ global gauge symmetry, $SU(2)$ spin symmetry, and particle-hole (geometric) symmetry of the Hubbard model.
8. Discuss solutions of the Hubbard model in the non-interacting and in the atomic limits.
9. Discuss the exact solution of the Hubbard model for two lattice sites.
10. Present the basic steps in deriving the effective $t - J$ Hamiltonian.
11. Discuss three theorems about the Hubbard model and their physical significance (no proofs).
12. Present the main steps in deriving the dynamical mean-field theory equations. Why it is an exact solution in the infinite dimension?

18 Literature

- H. Bruus, K. Flensberg, *Many-Body Quantum Theory in Condensed Matter Physics.*
- A. Altland, B. Simons, *Condensed Matter Field Theory.*
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- A.A Abrikosov, L..P. Gorkov, I.E. Dzyaloshinski, *Methods of Quantum Field Theory in Statistical Physics.*
- W. Nolting, *Fundamentals of Many-Body Physics.*
- R. A. Jishi, *Feynman Diagram Techniques in Condensed Matter Physics.*
- E. Fradkin, *Field Theories of Condensed Matter Physics.*
- N. Nagaosa, *Quantum Field Theory in Strongly Correlated Electronic Systems.*
- J. Spałek, *Wstęp do fizyki materii skondensowanej*, PWN.
- A. Georges, G. Kotliar, W. Krauth, and M.J. Rozenberg, *Dynamical mean-field theory of strongly correlated fermion systems and the limit of infinite dimensions*, Rev. Mod. Phys. 68, 13 (1996).
- More to be added in the course.