Topics in Many Body Theory

summer term 2024-25

Krzysztof Byczuk

Institute of Theoretical Physics, Faculty of Physics, University of Warsaw
byczuk@fuw.edu.pl
`https://www.fuw.edu.pl/~byczuk/`

Paweł Jakubczyk

Institute of Theoretical Physics, Faculty of Physics, University of Warsaw
pjak@fuw.edu.pl
`https://www.fuw.edu.pl/~pjak/`

11-04-2025

Rules

- Lectures and Tutorials are in Tuesdays at 11:15-13:00 in 1.02 room and in Thursdays at 8:15-10:00 in 1.01 room.
- Way of passing the course: There is an oral exam based on the theory from lectures (listed in this file) and problems solved on tutorials and homeworks (listed in this file). You are allowed to posses your handwritten notes from lectures and with solved problems during your exam and supposed to present 2-3 given topics in concise and short forms.

1 Week I, 24/02-02/03/2025

1.1 Lecture PJ

Review of the quantum many body formalism, notion or bosonic and fermionic coherent states, general expression for bosonic coherent states.

1.2 Tutorial KB

1. *Feynman path integral in quantum mechanics* - Express a propagator in one-particle quantum mechanics in terms of a path integral with a classical action
$$G(x,t;x',t') = \int \mathcal{D}[x(t)] e^{\frac{i}{\hbar}\int_t^{t'} d\tau \left(\frac{1}{2}m\dot{x}(t)^2 - V(x)\right)}$$
$$= \int \mathcal{D}[x(t)] e^{\frac{i}{\hbar}S[x(\tau)]}.$$
Discuss mathematical meaning of this functional integration.
2. *Green function (propagator) for free particle* - Derive an expression for the Green function (propagator) of one-particle in one dimension with $V(x) = 0$. Do it in two ways, directly from the definition in quantum mechanics and from the path integration in discrete version.

1.3 Homework problems

1. Demonstrate the closure relation $\int \prod_\alpha \frac{d\phi_\alpha^* d\phi_\alpha}{2\pi i} e^{-\sum_{\alpha'} \phi^*_{\alpha'}\phi_{\alpha'}} |\phi\rangle\langle\phi|$ for bosonic coherent states.
2. Consider $\langle \hat{N} \rangle = \frac{\langle\phi|\hat{N}|\phi\rangle}{\langle\phi|\phi\rangle}$, where $|\phi\rangle$ is a bosonic coherent state, and $\hat{N}$ denotes the total particle number operator. Express $\sigma^2 = \langle \hat{N}^2\rangle - \langle\hat{N}\rangle^2$ by the eigenvalues $\{\phi_\alpha\}$ corresponding to $|\phi\rangle$.
3. (Not obligatory) Consider the perturbed Gaussian measure $\frac{1}{Z(\lambda)} e^{-A(\vec{x},\lambda)}$. Notation follows Zinn-Justin's book and $V(\vec{x}) = \frac{1}{4!}\sum_{i=1}^N x_i^4$. Draw the Feynman graphs corresponding to the two-point function $\langle x_{i_1 i_2}\rangle_\lambda$ up to order λ^2. Use the given fact (linked cluster theorem) that the non-connected contributions cancel. Write down the analytical expressions corresponding to the diagrams and try to figure out the corresponding coefficients from combinatorial arguments.
4. *Quamtum mechanical operators and path integrals* - For a given trajectory we can define a classical momentum $m\dot{x}(t)$. Let's define at t_f a quantum mechanical momentum operator in one dimension as
$$\hat{p}\Psi(x_f,t_f) =$$
$$= \int_{-\infty}^{\infty} dy \int \mathcal{D}[x(t)] m\dot{x}(t_f) e^{\frac{i}{\hbar}S[x(\tau)]}\Psi(y,t_i).$$
Show that $\hat{p} = -i\hbar d/dx$.

2 Week II, 03-09/03/2024

2.1 Lecture PJ

Bosonic coherent states, Grassmann variables (basic notions and properties, conjugation, differentiation, integration), fermionic coherent states.

2.2 Tutorial KB

1. *Statistical mechanics and path integrals* - Formulation of the partition function in canonical ensemble in terms of the path integral
$$Z = \mathrm{Tr} e^{-\beta\hat{H}} =$$
$$= \int_{x(0)=x(\beta)} \mathcal{D}[x(\tau)] e^{-\frac{1}{\hbar}S[x(\tau)]},$$
where
$$S[x(\tau)] = \int_0^\beta d\tau \left(\frac{1}{2m}\left(\frac{dx(\tau)}{d\tau}\right)^2 + V(x(\tau))\right).$$
2. *Partition function for a harmonic oscillator* - Derive the partition function for a harmonic oscillator in quantum mechanics, i.e. $V(x) = \frac{1}{2}m\omega^2 x^2$. To compute the path integral use the Fourier series expansion in Matsubara frequencies $\omega_n = 2n\pi/\beta$ and the integration measure constant find from the free particle problem.

2.3 Homework problems

1. *2-point correlation functions for harmonic oscillator* - Two point correlation function is defined

$$G(\tau) = \frac{1}{Z}\mathrm{Tr}\left[e^{-\beta\hat{H}}\hat{x}(\tau)\hat{x}(0)\right],$$

where

$$\hat{x}(\tau) = e^{\frac{\hat{H}\tau}{\hbar}}\hat{x}e^{-\frac{\hat{H}\tau}{\hbar}},$$

with $0 \leqslant \tau < \beta\hbar$. The path integral representation of this correlation function is

$$G(\tau) = \frac{1}{Z}\int_{x(0)=x(\beta)} \mathcal{D}[x(\tau)]\ x(\tau)x(0)\ e^{-\frac{1}{\hbar}S[x(\tau)]}.$$

Determine this functional integral using the Fourier series expansion in Matsubara frequencies.

2. Show by direct means based on standard quantum statistical mechanics that the 2-point correlation function in harmonic oscillator takes the form

$$G(\tau) = \frac{\hbar}{2m\omega}\frac{\cosh\left[\left(\frac{\beta\hbar}{2}-\tau)\omega\right)\right]}{\sinh\left[\frac{\beta\hbar\omega}{2}\right]}.$$

Evaluate in details all steps needed in the problem 3 of the last tutorial.

3 Week III, 10-16/03/2025

3.1 Lecture PJ

Fermionic coherent states ctd: closure relation, trace formula. Coherent states for fermions and bosons summary. Propagator in QM (introduction)."

3.2 Tutorial KB

1. *Generating function* - For an arbitrary measure (probability distribution) in real n-dimensional space introduce the generating function for correlation functions. Check it properties.

2. *Gaussian integrals in n-dimensions* - Find the value of a gaussian integral of real variables in n-dimensional space.

3. *General gaussian integral* - Find the value of a gaussian integral with a linear term of real variables in n-dimensional space.

4. *Reminder of Grassmann algebra* - generators, anticommutation rules, reflection authomorphism P, even and odd Grassmann subspaces.

5. *Formal definitions* of conjugation, differentiation and integration in Grassmann algebra.

6. *Grassmann calculus* Let $f(\xi) = f_0 + f_1\xi$ and $A(\xi^*,\xi) = a_0 + a_1\xi + \bar{a}_1\xi^2 + a_{12}\xi^*\xi$. Compute

   (a) $\partial A(\xi^*,\xi)/\partial\xi$
   (b) $\partial A(\xi^*,\xi)/\partial\xi^*$
   (c) $\partial^2 A(\xi^*,\xi)/\partial\xi^*\partial\xi$
   (d) $\int d\xi f(\xi)$
   (e) $\int d\xi A(\xi^*,\xi)$
   (f) $\int d\xi^* A(\xi^*,\xi)$
   (g) $\int d\xi^* d\xi A(\xi^*,\xi)$

7. *Grassmann delta Dirac distribution* Show that $\delta(\xi,\xi) = \int \eta \exp(-\eta(\xi-\xi'))$ represents Dirac delta function.

3.3 Homework problems

1. Show that

$$\langle x_{k_1}x_{k_2}...x_{k_n}\rangle = \frac{\partial}{\partial b_{k_1}}\frac{\partial}{\partial b_{k_2}}...\frac{\partial}{\partial b_{k_n}}Z(\vec{b})|_{\vec{b}=0},$$

where $Z(\vec{b}) = \int d_n x e^{-\vec{b}\cdot\vec{x}}$.

2. Find explicit expression for

$$Z(\mathbf{A},\vec{b}) = \int d_n x e^{\frac{1}{2}\sum_{i,j=1}^{n} x_i A_{ij} x_j + \sum_{i=1}^{n} x_i b_i},$$

where A_{ij} is positive real symmetric matrix and b_i a real vector.

3. Review the demonstration of the closure relation using fermionic coherent states: $1 = \int\prod_\alpha d\xi_\alpha^* d\xi_\alpha e^{-\sum_\alpha \xi_\alpha^*\xi_\alpha}|\xi\rangle\langle\xi|$. Evaluate the final integral left out in the proof given in the lecture.

4 Week IV, 17-23/03/2025

4.1 Lecture PJ

Partition function and propagator in imaginary time for a single quantum particle; the classical limit. Coherent state path integral for the partition function of many-body systems. Evaluation of the coherent path integral for the partition function of noninteracting Bose and Fermi systems.

4.2 Tutorial KB

1. *Grassmann scalar product* Check that $\langle f|g\rangle = \int d\xi d\xi^* \exp(-\xi\xi^*)f^*(\xi^*)g(\xi)$ represents a natural scalar product between functions f and g.

2. Grassmann-Gaussian one-variable integral Show that $\int d\xi^* d\xi \exp(-\xi^* a\xi) = a$. Compare this result with Gaussian integral over complex numbers.

3. Change of variable From $\int \eta\eta = 1$ and by changing the variable $\eta = \alpha\xi$ show for a one Grassmann variable that the Jacobian $d(\alpha\xi) = (d\xi)/\alpha$. Apply this result to compute the Gaussian integral $\int d\xi^* d\xi \exp(-\xi^* a\xi) = a$ again.

4. *Evaluation of Matsubara sums* - Derive the contour integral equations for fermionic (F) and bosonic (B) Matsubara sums:

$$S^F(\tau) = \frac{1}{\beta}\sum_{ik_n} g(ik_n)e^{ik_n\tau} = -\oint_C \frac{dz}{2\pi i} n_F(z)g(z)e^{z\tau},$$

$$S^B(\tau) = \frac{1}{\beta}\sum_{i\omega_n} g(i\omega_n)e^{i\omega_n\tau} = \oint_C \frac{dz}{2\pi i} n_B(z)g(z)e^{z\tau},$$

where $k_n = (2n+1)\pi/\beta$, $\omega_n = 2n\pi/\beta$, and $n_{F(B)}(z) = 1/(e^{\beta z} \pm 1)$.

5. *Matsubara sum over functions with singe poles* - Derive a fermionic Matsubara sum for a function with single poles

$$g_0(z) = \prod_j \left(\frac{1}{z - z_j}\right).$$

Apply this result to the fermionic non-interacting Green function

$$G^0_{\mathbf{k}}(ik_n) = \frac{1}{ik_n - (\epsilon_{\mathbf{k}} - \mu)},$$

and derive probability of occupation of a single state $\mathbf{k}$.

4.3 Homework problems

1. *Normal ordering* - Consider a single quantum particle governed by the hamiltonian $H = \frac{\vec{p}^2}{2m} + V(\vec{x})$. Write down the complete (double series) expression for $: e^{-i\frac{H}{\hbar}\epsilon} :$. Compare it with the corresponding expression for $e^{-i\frac{H}{\hbar}\epsilon}$. Compare in particular the two expressions, when truncated at order ϵ and at order ϵ^2.

2. Show that

$$\int \prod_{i=1}^{n} d\eta_i^* d\eta_i e^{-\sum_{ij}\eta_i^* H_{ij}\eta_j + \sum_i \xi_i^*\eta_i + \xi_i\eta_i^*} =$$

$$= [\mathrm{Det}H] e^{\sum_{ij}\xi_i^*(H^{-1})_{ij}\xi_j}.$$

5 Week V, 24-30/03/2025

5.1 Lecture PJ

Real-time Green's functions, thermal (imaginary time) Green's functions and their path integral representation.

5.2 Tutorial KB

1. *Basic bosonic model* - The Hamiltonian is $\hat{H} = \epsilon\hat{b}^\dagger\hat{b}$, where $[\hat{b}, \hat{b}^\dagger] = 1$. Determine
   - Partition function directly from the trace
   - Partition function form the path integral and Matsubara-Fourier transform
   - Imaginary time Green's function from the equation of motion
   - Imaginary time Green's function from the path integral

5.3 Homework problems

1. *Basic fermionic model* - The Hamiltonian is $\hat{H} = \epsilon\hat{f}^\dagger\hat{f}$, where $\{\hat{f}, \hat{f}^\dagger\} = 1$. Determine
   - Partition function directly from the trace
   - Partition function form the path integral and Matsubara-Fourier transform
   - Imaginary time Green's function from the equation of motion
   - Imaginary time Green's function from the path integral

6 Week VI, 31/03-06/04/2025

6.1 Lecture PJ

Thermal Green's functions for non-interacting fermions and bosons, perturbation series for the partition function, Wick's theorem.

6.2 Tutorial KB

1. *Polaron model in the atomic limit* - Find the partition function of the polaron model in the atomic limit

$$\hat{H} = g\sum_\sigma(\hat{b}^\dagger + \hat{b})\hat{n}_\sigma - \sum_\sigma \mu_\sigma\hat{n}_\sigma + w\hat{b}^\dagger\hat{b}$$

by computing the Feynman path integral. $\hat{n}_\sigma = \hat{f}^\dagger_\sigma\hat{f}_\sigma$, $[\hat{b}, \hat{b}^\dagger] = 1$, $\{\hat{f}_\sigma, \hat{f}^\dagger_\sigma\} = 1$.

6.3 Homework problems

1. By direct calculation find the inverse of the matrix $S^{(\alpha)}$ discussed in the lecture and yielding the expression for the thermal Green's function of the non-interacting Fermi or Bose system.
   Hint: It may be helpful to perform the calculation for a small M (e.g. $M = 4$) first.

7 Week VII, 07-13/04/2025

7.1 Lecture PJ

Perturbation theory for the partition function and the grand canonical potential of interacting Fermi and Bose systems, Feynman diagrams, linked cluster theorem - proof by replica method.

7.2 Tutorial KB

1. Matsubara sum with a function possessing a branch cut

2. Wannier functions

3. derivation of the Hubbard model

7.3 Homework problems

1. A system of quantum particles (fermions or bosons) is described by the (grand-canonical) hamiltonian $H = H_0 + V$, where $H_0 = \sum_\alpha (\epsilon_\alpha - \mu) a^\dagger_\alpha a_\alpha$, and V represents a two-body interaction. Consider perturbative expansion for the partition function Z up to second order in V.
   a) How many terms (contractions) are there?
   b) Draw all the distinct Feynman diagrams (there are 10) and figure out the corresponding numerical coefficients. Check consistency by comparing with point a). c) Which diagrams obtained in point b) contribute to the grand canonical potential Ω?

8 Literature

- H. Bruus, K. Flensberg, *Many-Body Quantum Theory in Condensed Matter Physics.*
- A. Altland, B. Simons, *Condensed Matter Field Theory.*
- J.W. Negele, H. Orland, *Quantum many-paricle systems.*
- J. Zinn-Justin, *Phase Transitions and Renormalization Group.*
- R.D. Mattuck, *A guide to Feynman diagrams in the many-body problems.*
- A.A Abrikosov, L..P. Gorkov, I.E. Dzyaloshinski, *Methods of Quantum Field Theory in Statistical Physics.*
- W. Nolting, *Fundamentals of Many-Body Physics.*
- R. A. Jishi, *Feynman Diagram Techniques in Condensed Matter Physics.*
- E. Fradkin, *Field Theories of Condensed Matter Physics.*
- N. Nagaosa, *Quantum Field Theory in Strongly Correlated Electronic Systems.*
- J. Spałek, *Wstęp do fizyki materii skondensowanej*, PWN.
- A. Georges, G. Kotliar, W. Krauth, and M.J. Rozenberg, *Dynamical mean-field theory of strongly correlated fermion systems and the limit of infinite dimensions*, Rev. Mod. Phys. 68, 13 (1996).
- More to be added in the course.