Two identical bosons

$V=\frac{1}{2}m\omega^2 x^2$

$E_g = \langle 0|\hat{H}|0\rangle$ – 1st order perturbation

$V(x_1,x_2) = \alpha e^{-\beta(x_1-x_2)^2}$

$\beta > 0$

$$\hat{H} = \hat{H}_0 + \hat{H}' = \sum_{i=1}^{2}\left(\frac{\hat{p}_i^2}{2m} + \frac{1}{2}m\omega^2 x_i^2\right) + V(x_1 - x_2)$$

$$E_g = E_0 + \Delta E$$

$$E_0 = \langle 0|\hat{H}_0|0\rangle = \int dx_1 \int dx_2\, \Psi_0^*(x_1,x_2)\, H_0\, \Psi_0(x_1,x_2)$$

$\Psi_0(x_1,x_2)$ – a wave function of 2 bosons in GS

Ground state of Harmonic oscillator

$$\varphi_0(x) = \sqrt[4]{\frac{a^2}{\pi}}\, e^{-\frac{a^2x^2}{2}} \quad , \quad a = \sqrt{\frac{m\omega}{\hbar}}$$

$\int dx\, |\varphi_0(x)|^2 = 1$

$[a] = \left[\frac{1}{m}\right]$

$[\varphi_0] = \left[\frac{1}{\sqrt{m}}\right]$

$$\Psi_0(x_1,x_2) = \hat{S}_B\, \varphi_0(x_1)\varphi_0(x_2) =$$

$$= \frac{1}{2}\left(\varphi_0(x_1)\varphi_0(x_2) + \varphi_0(x_2)\varphi_0(x_1)\right) = \varphi_0(x_1)\varphi_0(x_2)$$

two bosons in GS

$$E_0 = \int dx_1 \int dx_2\, \varphi_0^*(x_1)\varphi_0^*(x_2)\left[\frac{\hat{p}_1^2}{2m} + \frac{1}{2}m\omega x_1^2 + \frac{\hat{p}_2^2}{2m} + \frac{1}{2}m\omega x_2^2\right]\varphi_0(x_1)\varphi_0(x_2) =$$

$$= \frac{1}{2}\hbar\omega + \frac{1}{2}\hbar\omega = \hbar\omega.$$

$$\Delta E = \int dx_1 \int dx_2\, \varphi_0^*(x_1)\varphi_0^*(x_2)\, V(x_1 - x_2)\, \varphi_0(x_1)\varphi_0(x_2) =$$

$$= \frac{a^2}{\pi}\alpha \int dx_1 \int dx_2\, e^{-a^2x_1^2 - a^2x_2^2}\, e^{-\beta(x_1-x_2)^2}$$

changing variables

$$\begin{cases} y_1 = \dfrac{x_1+x_2}{2} \\ y_2 = \dfrac{x_2-x_1}{2} \end{cases} \rightarrow \begin{cases} x_2 = y_1+y_2 \\ x_1 = y_1-y_2 \end{cases}$$

$$x_1^2+x_2^2 = (x_1+x_2)^2-2x_1x_2 = 4y_1^2-2y_1^2+2y_2^2 = 2(y_1^2+y_2^2)$$

$$(x_1-x_2)^2 = 4y_2^2$$

$$J = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 2$$

$$\Delta E = 2\frac{a^2}{\pi}\alpha \int dy_1 \int dy_2 \, e^{-2a^2y_1^2} e^{-(4\beta+2a^2)y_2^2} =$$

$$= 2\frac{a^2}{\pi}\alpha\sqrt{\frac{\pi}{2a^2}}\sqrt{\frac{\pi}{4\beta+2a^2}} = \frac{a\,\alpha}{\sqrt{2\beta+a^2}}$$

$$\boxed{E_g = \hbar\omega + \frac{a\,\alpha}{\sqrt{2\beta+a^2}}}$$

$[\hbar\omega] = [J \cdot s \cdot \frac{1}{s}] = [J]$

$[\alpha] = [J]$

$[a] = [\frac{1}{m}]$ $\qquad [E_g] = [J]$

$[\beta] = [\frac{1}{m^2}]$