1^{st} excited state

$$\left\{ \left| \begin{matrix} \uparrow \\ \uparrow \end{matrix} \right\rangle, \left| \begin{matrix} \downarrow \\ \downarrow \end{matrix} \right\rangle, \left| \begin{matrix} \downarrow \\ \uparrow \end{matrix} \right\rangle, \left| \begin{matrix} \uparrow \\ \downarrow \end{matrix} \right\rangle \right\} = \{|1\rangle, |2\rangle, |3\rangle, |4\rangle\}$$

$\hat{H}'$ - spin independent interaction

$\langle x|n\rangle\langle\sigma|\tau\rangle = \varphi_n(x)\delta_{\sigma\tau}$

$\parallel$

$\langle x\sigma|n\tau\rangle = \varphi_{n\tau}(x\sigma)$

$\leftarrow$ pro $\quad$ $\llcorner$ wavefunction

$$\langle i|\hat{H}'|j\rangle = \begin{pmatrix} \langle 1|H'|1\rangle & \langle 1|H'|2\rangle & \dots \\ \langle 2|H'|1\rangle & \langle 2|H'|2\rangle & \\ \langle 3|H'|1\rangle & \langle 3|H'|2\rangle & \\ \langle 4|H'|1\rangle & \langle 4|H'|2\rangle & \end{pmatrix}$$

$$|1\rangle = \left| \begin{matrix} \uparrow \\ \uparrow \end{matrix} \right\rangle = \frac{1}{\sqrt{2}}\left[|1\uparrow\rangle_1 \otimes |2\uparrow\rangle_2 - |2\uparrow\rangle_1 \otimes |1\uparrow\rangle_2 \right]$$

$$\Psi^1 = \Psi_{1\uparrow 2\uparrow}(x_1\sigma_1; x_2\sigma_2) = \langle x_1\sigma_1; x_2\sigma_2|1\rangle =$$

$$= {}_1\langle x_1\sigma_1| \otimes {}_2\langle x_2\sigma_2| \frac{1}{\sqrt{2}}\left[|1\uparrow\rangle_1 \otimes |2\uparrow\rangle_2 - |2\uparrow\rangle_1 \otimes |1\uparrow\rangle_2 \right] =$$

$$= \varphi_1(x_1)\delta_{\sigma_1\uparrow}\varphi_2(x_2)\delta_{\sigma_2\uparrow} - \varphi_2(x_1)\delta_{\sigma_1\uparrow}\varphi_1(x_2)\delta_{\sigma_2\uparrow} =$$

$$= \left(\varphi_1(x_1)\varphi_2(x_2) - \varphi_2(x_1)\varphi_1(x_2)\right)\delta_{\sigma_1\uparrow}\delta_{\sigma_2\uparrow} =$$

$$= \left(\varphi_1(x_1)\varphi_2(x_2) - \varphi_2(x_1)\varphi_1(x_2)\right)\begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix} =$$

$$= \left(\varphi_1(x_1)\varphi_2(x_2) - \varphi_2(x_1)\varphi_1(x_2)\right) \cdot \begin{pmatrix}1\\0\\0\\0\end{pmatrix}$$

$$\Psi^2 = \Psi_{1\downarrow 2\downarrow}(x_1\sigma_1\, x_2\sigma_2) = \langle x_1\sigma_1; x_2\sigma_2|2\rangle =$$

$$= {}_1\langle x_1\sigma_1| {}_2\langle x_2\sigma_2| \cdot \frac{1}{\sqrt{2}}\left[|1\downarrow\rangle_1 |2\downarrow\rangle_2 - |2\downarrow\rangle_1 |1\downarrow\rangle_2 \right] =$$

$$= \left(\varphi_1(x_1)\varphi_2(x_2) - \varphi_2(x_1)\varphi_1(x_2)\right)\delta_{\sigma_1\downarrow}\delta_{\sigma_2\downarrow} =$$

$$= \left(\varphi_1(x_1)\varphi_2(x_2) - \varphi_2(x_1)\varphi_1(x_2)\right)\begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix} =$$

$$= \left(\varphi_1(x_1)\varphi_2(x_2) - \varphi_2(x_1)\varphi_1(x_2)\right)\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

$$\mathcal{H} = L^2(\mathbb{R}) \otimes S_{1/2}$$

$$|\psi\rangle = |n\tau\rangle = |n\rangle \otimes |\tau\rangle$$

$$\psi_{n\tau}(x\sigma) = \langle x\sigma|\psi\rangle = \langle x| \otimes \langle\sigma|\psi\rangle =$$

$$= \langle x|n\rangle\langle\sigma|\tau\rangle = \psi_n(x)\,\delta_{\sigma\tau}$$

$$|3\rangle = \frac{1}{\sqrt{2}}\left[\,|1\uparrow\rangle_1 |2\downarrow\rangle_2 - |2\downarrow\rangle_1 |1\uparrow\rangle_2\,\right]$$

$$\Psi^3_{1\uparrow 2\downarrow}(x_1\sigma_1\, x_2\sigma_2) = \frac{1}{\sqrt{2}}\left[\phi_1(x_1)\delta_{\sigma_1\uparrow}\,\phi_2(x_2)\delta_{\sigma_2\downarrow} - \phi_2(x_1)\delta_{\sigma_1\downarrow}\,\phi_1(x_2)\delta_{\sigma_2\uparrow}\right] =$$

$$= \frac{1}{\sqrt{2}}\left[\phi_1(x_1)\phi_2(x_2)\begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}0\\1\end{pmatrix} - \phi_2(x_1)\phi_1(x_2)\begin{pmatrix}0\\1\end{pmatrix}\otimes\begin{pmatrix}1\\0\end{pmatrix}\right] =$$

$$= \frac{1}{\sqrt{2}}\left[\phi_1(x_1)\phi_2(x_2)\begin{pmatrix}0\\1\\0\\0\end{pmatrix} - \phi_2(x_1)\phi_1(x_2)\begin{pmatrix}0\\0\\1\\0\end{pmatrix}\right]$$

$$|4\rangle = \frac{1}{\sqrt{2}}\left[\,|1\downarrow\rangle_1 |2\uparrow\rangle_2 - |2\uparrow\rangle_1 |1\downarrow\rangle_2\,\right]$$

$$\Psi^4_{1\downarrow 2\uparrow} = \frac{1}{\sqrt{2}}\left[\phi_1(x_1)\delta_{\sigma_1\downarrow}\,\phi_2(x_2)\delta_{\sigma_2\uparrow} - \phi_2(x_1)\delta_{\sigma_1\uparrow}\,\phi_1(x_2)\delta_{\sigma_2\downarrow}\right] =$$

$$= \frac{1}{\sqrt{2}}\left[\phi_1(x_1)\phi_2(x_2)\begin{pmatrix}0\\1\end{pmatrix}\otimes\begin{pmatrix}1\\0\end{pmatrix} - \phi_2(x_1)\phi_1(x_2)\begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}0\\1\end{pmatrix}\right] =$$

$$= \frac{1}{\sqrt{2}}\left[\phi_1(x_1)\phi_2(x_2)\begin{pmatrix}0\\0\\1\\0\end{pmatrix} - \phi_2(x_1)\phi_1(x_2)\begin{pmatrix}0\\1\\0\\0\end{pmatrix}\right]$$

Matrix elements

$$\langle i|H'|j\rangle = \int dx_1 \int dx_2 \sum_{\sigma_1\sigma_2} \Psi^i(x_1,x_2)^*\, V(x_1,x_2)\, \Psi^j(x_1,x_2)$$

spin independent

$$\langle 1|H'|1\rangle \sim (1\,0\,0\,0)\begin{pmatrix}1\\0\\0\\0\end{pmatrix} = 1$$

$$\langle 2|H'|2\rangle \sim (0\,0\,0\,1)\begin{pmatrix}0\\0\\0\\1\end{pmatrix} = 1$$

$$\langle 1|H'|2\rangle \sim (1\,0\,0\,0)\begin{pmatrix}0\\0\\0\\1\end{pmatrix} = 0$$

$$\langle 2|H'|1\rangle \sim (0\,0\,0\,1)\begin{pmatrix}1\\0\\0\\0\end{pmatrix} = 0$$

$$\langle 1|H'|1\rangle = \langle 2|H'|2\rangle = \frac{1}{2}\int dx_1 \int dx_2 \left[\phi_1(x_1)\phi_2(x_2) - \phi_2(x_1)\phi_1(x_2)\right]^* \overbrace{\cdot}^{V(x_1-x_2)} \left[\phi_1(x_1)\phi_2(x_2) - \phi_2(x_1)\phi_1(x_2)\right] =$$

$$= \frac{1}{2}\int dx_1 \int dx_2\, V(x_1-x_2)\Big[|\phi_1(x_1)|^2|\phi_2(x_2)|^2 + |\phi_2(x_1)|^2|\phi_1(x_2)|^2 - \phi_1^*(x_1)\phi_2^*(x_2)\phi_2(x_1)\phi_1(x_2) - \phi_2^*(x_1)\phi_1^*(x_2)\phi_1(x_1)\phi_2(x_2)\Big]$$

$$\langle 3|H'|3\rangle = \frac{1}{2}\int dx_1 \int dx_2\, V(x_1, x_2) \left[\varphi_1^*(x_1)\,\varphi_2^*(x_2)\,(0\,1\,0\,0) - \varphi_2^*(x_1)\,\varphi_1^*(x_2)\,(0\,0\,1\,0) \right] \cdot$$

$$\left[\varphi_1(x_1)\,\varphi_2(x_2) \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} - \varphi_2(x_1)\,\varphi_1(x_2) \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right] =$$

$$= \frac{1}{2} \int dx_1\, dx_2\, V(x_1, x_2) \left[\varphi_1^*(x_1)\,\varphi_2^*(x_2)\,\varphi_1(x_1)\,\varphi_2(x_2) + \varphi_2^*(x_1)\,\varphi_1^*(x_2)\,\varphi_2(x_1)\,\varphi_1(x_2) \right] =$$

$$= \frac{1}{2} \int dx_1\, dx_2\, V(x_1, x_2) \left[|\varphi_1(x_1)|^2 |\varphi_2(x_2)|^2 + |\varphi_2(x_1)|^2 |\varphi_1(x_2)|^2 \right]$$

$$\langle 4|H'|4\rangle = \frac{1}{2}\int dx_1\, dx_2\, V(x_1 - x_2) \left[\varphi_1^*(x_1)\,\varphi_2^*(x_2)\,(0\,0\,1\,0) - \varphi_2^*(x_1)\,\varphi_1^*(x_2)\,(0\,1\,0\,0) \right]$$

$$\left[\varphi_1(x_1)\,\varphi_2(x_2) \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} - \varphi_2(x_1)\,\varphi_1(x_2) \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right] =$$

$$= \frac{1}{2} \int dx_1\, dx_2\, V(x_1 - x_2) \left[\varphi_1^*(x_1)\,\varphi_2^*(x_2)\,\varphi_1(x_1)\,\varphi_2(x_2) + \varphi_2^*(x_1)\,\varphi_1^*(x_2)\,\varphi_2(x_1)\,\varphi_1(x_2) \right] = \langle 3|H'|3\rangle$$

$$\langle 3|H'|4\rangle = \frac{1}{2}\int dx_1 \int dx_2\, V(x_1, x_2) \cdot$$

$$\left[\varphi_1^*(x_1)\,\varphi_2^*(x_2)\,(0\,1\,0\,0) - \varphi_2^*(x_1)\,\varphi_1^*(x_2)\,(0\,0\,1\,0) \right] \cdot$$

$$\left[\varphi_1(x_1)\,\varphi_2(x_2) \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} - \varphi_2(x_1)\,\varphi_1(x_2) \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right] =$$

$$= -\frac{1}{2} \int dx_1 \int dx_2\, V(x_1, x_2) \left[\varphi_1^*(x_1)\,\varphi_2^*(x_2)\,\varphi_2(x_1)\,\varphi_1(x_2) + \varphi_2^*(x_1)\,\varphi_1^*(x_2)\,\varphi_1(x_1)\,\varphi_2(x_2) \right] = \langle 4|H'|3\rangle^*$$

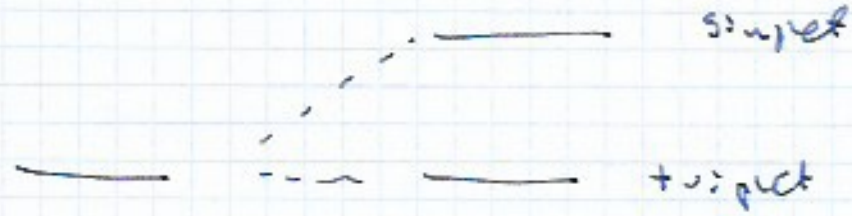
$$\det \left(\begin{array}{cc|cc} a-\lambda & 0 & \multicolumn{2}{c}{0} \\ 0 & a-\lambda & & \\ \hline \multicolumn{2}{c|}{0} & b-\lambda & c \\ & & c & b-\lambda \end{array} \right) = 0$$

$$(b-\lambda)^2 = c^2 \;\rightarrow\; \lambda = \pm c + b$$

$$\lambda_1 = \lambda_2 = \frac{1}{2}\int dx_1 \int dx_2\, V(x_1, x_2)\Big[\, |\phi_1(x_1)|^2 |\phi_2(x_2)|^2 + |\phi_2(x_1)|^2 |\phi_1(x_2)|^2 - \phi_1^*(x_1)\phi_2^*(x_2)\,\phi_2(x_1)\phi_1(x_2) - \phi_2^*(x_1)\phi_1^*(x_2)\,\phi_1(x_1)\phi_2(x_2)\Big]$$

$$\lambda_{3,4} = \frac{1}{2}\int dx_1 \int dx_2\, V(x_1, x_2)\Big[\, |\phi_1(x_1)|^2 |\phi_2(x_2)|^2 + |\phi_2(x_1)|^2 |\phi_1(x_2)|^2 \pm \phi_1^*(x_1)\phi_2^*(x_2)\,\phi_2(x_1)\phi_1(x_2) \pm \phi_2^*(x_1)\phi_1^*(x_2)\,\phi_1(x_1)\phi_2(x_2)\Big]$$

$$\lambda_1 = \lambda_2 = \lambda_3^- \neq \lambda_4^+$$



$$\begin{pmatrix} b - b \pm c & c \\ c & b - b \pm c \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 0$$

$$\begin{pmatrix} \pm 1 & 1 \\ 1 & \pm 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 0 \quad \Rightarrow \quad |u_-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad |u_+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\psi^s = \frac{1}{\sqrt{2}}\left(\psi^3_{1\uparrow 2\downarrow} + \psi^4_{1\downarrow 2\uparrow}\right) \qquad S_{tot} = 1,\ S_z = 0$$

$$\psi^a = \frac{1}{\sqrt{2}}\left(\psi^3_{1\uparrow 2\downarrow} - \psi^4_{1\downarrow 2\uparrow}\right) \qquad S_{tot} = 0,\ S_z = 0$$