Verschränkung II

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What you would like to know about

entanglement

but you were afraid to ask

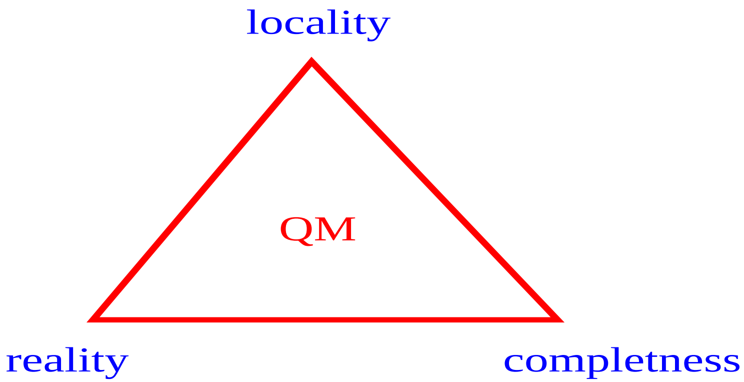
Main goal:

- Entanglement is a quantum correlation in quantum many body system
- Entanglement does not depend on particular physical representation
- Entanglement is a resource like energy
- Entanglement can be quantified and measured

Plan of the talk:

1. EPR and introduction to entanglement notion
2. How to use entanglement
    - quantum teleportation
3. How to entangle photons, electrons, ...experimentally
4. How to characterize entanglement
    - pure vs. mixed states entanglement
    - measures of entanglement
5. How to quantify correlations in bulk systems
6. Fermions and bosons
7. Conclusions and outlook: *correlations without correlata?*

EPR theorem today

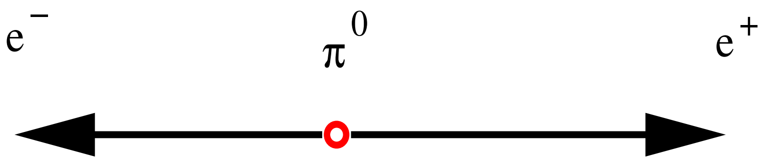


$$\mathcal{H} = \mathcal{H}_+ \otimes \mathcal{H}_-$$

$$|\Psi_{\mathrm{EPR}}\rangle = [|\uparrow\rangle_- \otimes |\downarrow\rangle_+ - |\downarrow\rangle_- \otimes |\uparrow\rangle_+]/\sqrt{2}$$

Verschränkung - entanglement (Schrödinger 1935)

QM is nonlocal



correlations over distance

results of independent measurements will be correlated

no superluminal transfer of information, energy, etc.

Bipartite pure entanglement

Let $\{|i\rangle_A \otimes |j\rangle_B\} \in \mathcal{H} = \mathcal{H}_\mathcal{A} \otimes \mathcal{H}_\mathcal{B}$ and AB distinguishable.

Any state

$$|\Psi\rangle = \sum_{ij} \gamma_{ij} |i\rangle_A \otimes |j\rangle_B$$

that cannot be represented as a product state is called an entangled state.

- Entanglement is a quantum correlation which does not have a classical counterpart
- any entangled state cannot be prepared from a product state by local operations and classical communications (LOCC)

Bell states

- classical two level system (0 or 1) codes one bit of information
- in QM two level system can be both 0 and 1 (spin, polarization, vortex, energy structure)
- it was proposed to call it quantum bit or qbit (read: *qiubit*) in general - Schumacher (1995)

Bell states - maximally entangled states of two qbits

$$|\Psi^-\rangle = \frac{1}{\sqrt{2}}\left[|01\rangle - |10\rangle\right]$$

$$|\Psi^+\rangle = \frac{1}{\sqrt{2}}\left[|01\rangle + |10\rangle\right]$$

$$|\Phi^-\rangle = \frac{1}{\sqrt{2}}\left[|00\rangle - |11\rangle\right]$$

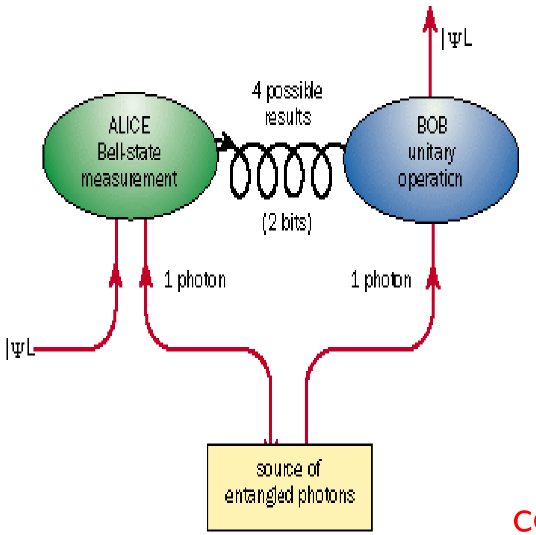
$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}\left[|00\rangle + |11\rangle\right]$$

Quantum teleportation

Bennett *et al.* (1993), photons (1998-2005), atoms (2004)

Alice and Bob share one entangled state, e.g. $|\Phi^+\rangle$.
Alice wants to send to Bob all necessary information about the unknown quantum state $|\Phi\rangle = a|0\rangle + b|1\rangle$ she has got such that Bob could recreate this state using a particle he has at hand. This is a task of quantum teleportation. The state at Alice will be destroyed.
What about the entangled state they share?

$$|\Phi\rangle|\Phi^+\rangle \sim [|\Phi^+\rangle(a|0\rangle + b|1\rangle) + |\Phi^-\rangle(a|0\rangle - b|1\rangle) + |\Psi^+\rangle(a|1\rangle + b|0\rangle) + |\Psi^-\rangle(a|1\rangle - b|0\rangle)]$$



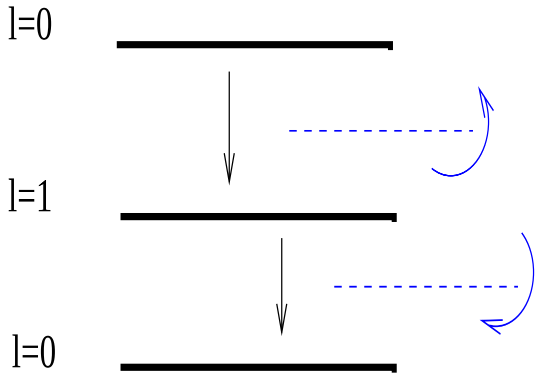
A: performs projective measurement on her 2 qbits - LO

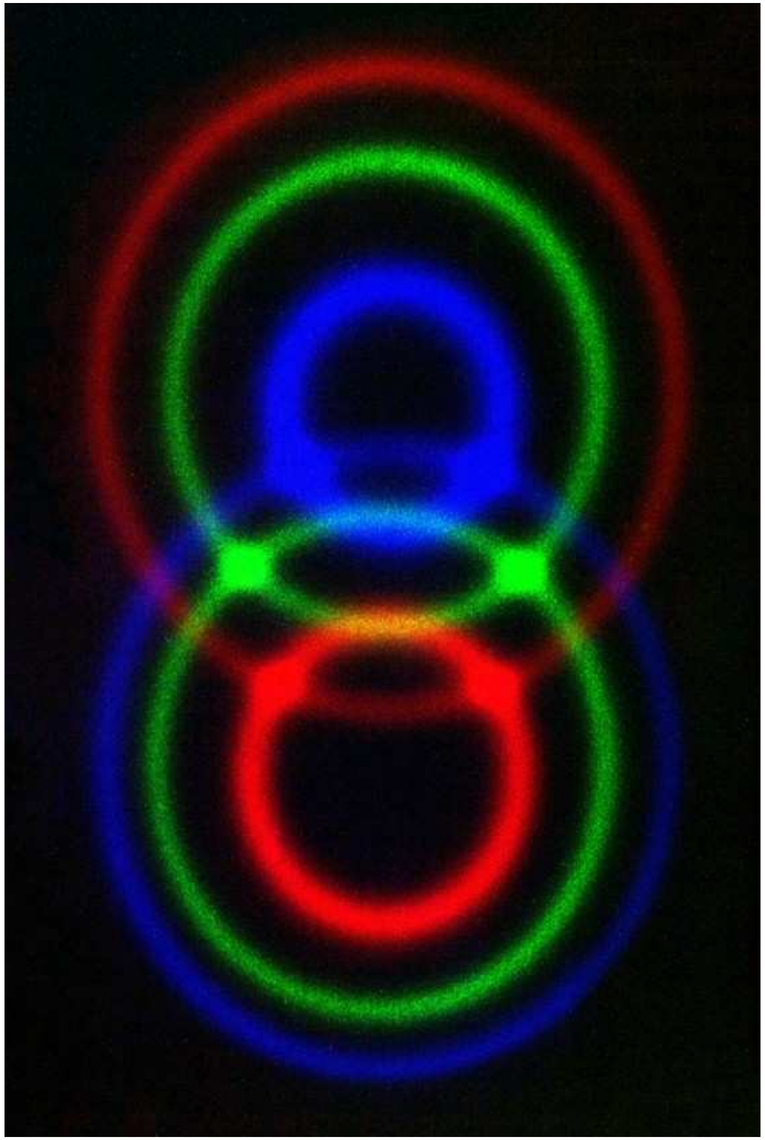
A: call Bob and tells her result (one of 4) - CC

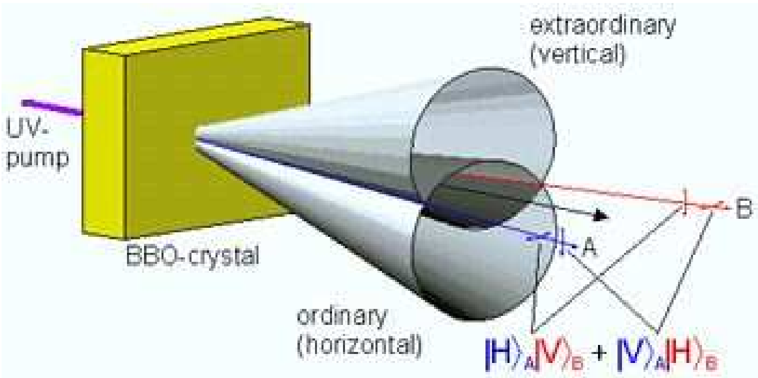
B: depending on A info performs 1 or σ_x or/and σ_z - LO

cost: one Bell state is eatten up

Source of entangled photons



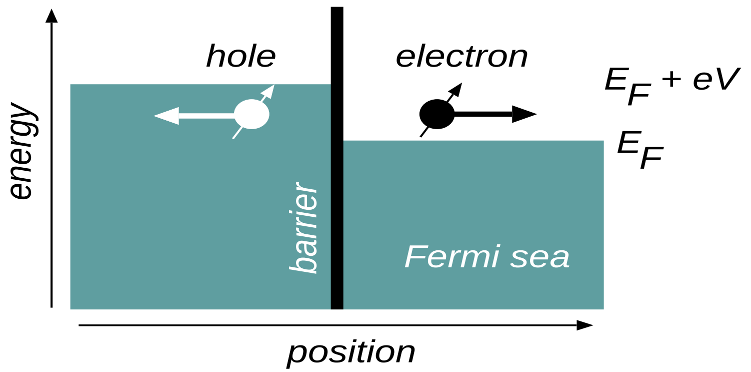


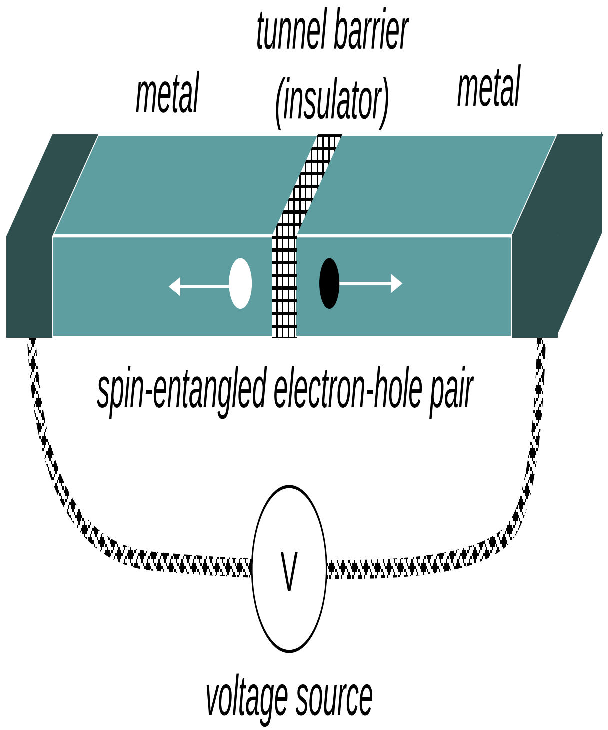


parametric down-conversion, Kwiat *et al.* (1995)

light emitting quantum dots, beam-splitters,...

Source of entangled fermions





Beenakker *et al.* (2003)

$$|\uparrow\rangle_e|\uparrow\rangle_h + e^{i\phi}|\downarrow\rangle_e|\downarrow\rangle_h$$

electron-electron scattering, quantum dots, Cooper pairs, Kondo scattering, ...

Mixed state

- density operator $\hat{\rho} = \sum_n p_n |\Psi_n\rangle\langle\Psi_n|$ describes a system coupled to another system to which we do not have an access
- pure state - maximal knowledge $\hat{\rho}^2 = \hat{\rho}$
- mixed state - statistical knowledge, mixture of different pure states can lead to the same density operator and thereby the same mixed state
- states from different ensembles having the same density operator are experimentally indistinguishable
- when pure system has entangled subsystems then each subsystem is in a mixed state, e.g.

$$|\Psi\rangle = \alpha|00\rangle + \beta|11\rangle$$

reduced density operator

$$\hat{\rho}_A = Tr_B\hat{\rho} = Tr_B|\Psi\rangle\langle\Psi| = |\alpha|^2|0\rangle\langle 0| + |\beta|^2|1\rangle\langle 1|$$

Positive definition: A pure bipartite quantum system is entangled if from the each subsystem point of view it looks as a mixed state

GHZ - entangled state

Greenberger-Horne-Zeilinger (GHZ)

$$|\Psi_{abc}\rangle = \frac{1}{\sqrt{2}} (|0_a 0_b 0_c\rangle + |1_a 1_b 1_c\rangle)$$

obviously this state is entangled

but

$$\hat{\rho}_{bc} = Tr_a |\Psi_{abc}\rangle\langle\Psi_{abc}| = \frac{1}{2} (|0_b 0_c\rangle\langle 0_b 0_c| + |1_b 1_c\rangle\langle 1_b 1_c|)$$

b and c are not entangled! (the same is true for ab and ac pairs)

If a makes a projective measurement on $|\pm\rangle = (|0\rangle \pm |1\rangle)/\sqrt{2}$ then

$$|\Psi^{\pm}_{bc}\rangle = \frac{1}{\sqrt{2}} (|0_b 0_c\rangle \pm |1_b 1_c\rangle)$$

is an entangled state; a must call the result to b and c using CC!

Entanglement in mixed state

A mixed state is not entangled if there exists a convex decomposition into pure product state of its density operator, i.e.

$$\hat{\rho} = \sum_n p_n |\Psi_n\rangle\langle\Psi_n|$$

with

$$|\Psi_n\rangle = |\Psi_n^A\rangle|\Psi_n^B\rangle$$

for each n.

$$\hat{\rho}_{sep} = \sum_n p_n \hat{\rho}_A \otimes \hat{\rho}_B$$

- mixture of separable states is always separable
- mixture of entangled states need not be entangled (see example)

Mixture of Bell states

$$\hat{\rho} = \frac{1}{2}|\Phi^+\rangle\langle\Phi^+| + \frac{1}{2}|\Phi^-\rangle\langle\Phi^-| = \frac{1}{2}|00\rangle\langle 00| + \frac{1}{2}|11\rangle\langle 11| =$$

$$\frac{1}{2}\left[|0\rangle\langle 0|_A \otimes |0\rangle\langle 0|_B + |1\rangle\langle 1|_A \otimes |1\rangle\langle 1|_B\right]$$

- the mixed state can be realized by both an ensemble of maximally entangled states and an ensemble of product states
- mixture is a process which destroys entanglement

Example: Werner state

$$\hat{\rho} = \frac{1}{4}(1-\lambda)\hat{Id} + \lambda|\Psi^-\rangle\langle\Psi^-|$$

is entangled for $|\lambda| > 1/3$.

Entanglement measures

- finite regime - for a single copy of a quantum state
- asymptotic regime - for n copies of a quantum state with $n \longrightarrow \infty$

Maximally entangled state in pure bipartite states $\mathcal{H}_d \otimes \mathcal{H}_d$

$$|\Phi_{max}\rangle = \sum_{i=1}^{d} \frac{1}{\sqrt{d}} |\phi_i\rangle \otimes |\phi_i\rangle$$

Entanglement measure $E(\hat{\rho})$ is a real-valued function

$$E: \quad \hat{\rho} \longrightarrow E(\hat{\rho}) \in R,$$

satisfying reasonable postulates:

Entanglement measures postulates

1. **separability**: $E(\hat{\rho}) = 0$ for $\hat{\rho}$ separable
2. **normalization**: $E(\hat{\rho}) = \log_2 d$ for maximally $|\Phi_{max}\rangle$ entangled state
3. **monotonicity**: $E(\hat{\Lambda}\hat{\rho}) \leqslant E(\hat{\rho})$ for any LOCC $\hat{\Lambda}$ [LOCC does not increase entanglement]
4. **continuity**: If $||\hat{\rho} - \hat{\sigma}|| \to 0$ then $E(\hat{\rho}) - E(\hat{\sigma}) \to 0$
5. **additivity**: $E(\hat{\rho}^{\otimes n}) = nE(\hat{\rho})$
6. **subadditivity**: $E(\hat{\rho} \otimes \hat{\sigma}) \leqslant E(\hat{\rho}) + E(\hat{\sigma})$
7. **regularization**: $E^{\infty}(\hat{\rho}) = lim_{n\to\infty} E(\hat{\rho}^{\otimes n})/n$ exists
8. **convexity**: $E(\lambda\hat{\rho} + (1-\lambda)\hat{\sigma}) \leqslant \lambda E(\hat{\rho}) + (1-\lambda)E(\hat{\sigma})$, for $0 \leqslant \lambda \leqslant 1$ [mixing does not increase entanglement]

Pure bipartite states

relative von Neumann entropy $(\hat{\rho} = |\Psi\rangle\langle\Psi|)$

$$E(|\Psi\rangle) = -Tr[\hat{\rho}_A \log_2 \hat{\rho}_A] = -Tr[\hat{\rho}_B \log_2 \hat{\rho}_B]$$

Schmidt rank r $(|\Psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B,\, dim\mathcal{H}_A \leqslant dim\mathcal{H}_B)$

$$|\Psi\rangle = \sum_{i=1}^{r} p_i |\tilde{\Psi}_i^A\rangle |\tilde{\Psi}_i^B\rangle$$

$r \leqslant dim\mathcal{H}_A$ - number of nonzero terms in Schmidt decomposition
(number of entangled degrees of freedom)

Pure bipartite states - example

Single qbit

$$|\phi\rangle = \sum_{ij=1}^{2} \gamma_{ij} |i\rangle_A |j\rangle_B$$

$Tr\gamma\gamma^+ = 1$

$$E(|\phi\rangle) = \mathcal{F}(\frac{1}{2} + \frac{1}{2}\sqrt{1 - 4\det\gamma\gamma^+})$$

where

$\mathcal{F}(x) = -x\log_2 x - (1-x)\log_2(1-x)$

concurrency

$$C = 2\sqrt{\det\gamma\gamma^+}$$

Bell state $\gamma = \sigma_x$
$C = 1$, $E = \mathcal{F}(1/2) = 1$, $r = 2$
Product state $|ii\rangle$
$C = 0$, $E = \mathcal{F}(0) = 0$, $r = 1$

Mixed bipartite states

not completely resolved because of the intricate relation between classical and quantum correlations in mixed states

- E_c - entanglement cost (minimal number of Bell states to create a given state using LOCC) [cont]
- E_D - entanglement of distillation (maximal number of Bell states extracted from a system using LOCC) [cont]
- E_F - entanglement of formation (optimized average von Neumann entropy of reduced density operators for pure states) [add]

$$E_F(\hat{\rho}) = \min_{\{p_i, |\Psi_i\rangle\}} \sum_i p_i S(\hat{\rho}_{i,red})$$

- E_R - relative entropy (distance between entangled $\hat{\rho}$ and the closest separated $\hat{\sigma}$) [add]

$$E_R(\hat{\rho}) = \min_{\hat{\sigma}} [Tr \hat{\rho}(\log_2 \hat{\rho} - \log_2 \hat{\sigma})]$$

  (quite useful)
- many others ...

$$E_D \leqslant E_F \leqslant E_C$$

Information and correlation (Shannon)

- One source of messages m_k with receiving them with probability p_k $(k = 1, \dots, d)$.
- Df. Information in a message m_k is $I(m_k) = -\log_2 p_k$.
- Df. Average information (surprise)

$$I = \langle I(m_k) \rangle - -\sum_{k=1}^{d} p_k \log_2 p_k.$$

  - $0 \leqslant I \leqslant \log_2 d$.
  - $I = \log_2 d$ if $p_k = 1/d$ for each k (equal probable).
  - Shannon's entropy (after von Neumann suggestion).

Information and correlation (Shannon)

- Two sources of messages x_m and y_n with probability distribution $p(x, y)$.
- Df. Total information in message (x, y) is

$$I(x, y) = -\log_2 p(x, y).$$

- Df. Average total information in average

$$I = \langle I(x, y) \rangle = -\sum_{x,y} p(x, y) \log_2 p(x, y)$$

- Df. Mutual information

$$\Delta I(p_1, p_2; p) = I_1 + I_2 - I \equiv I(p||p_1 p_2) \geqslant 0$$

  - where

$$I(p||p_1 p_2) \equiv -\sum_{x,y} p(x, y) \log_2 \frac{p(x, y)}{p_1(x) p_2(y)}$$

    can be called a relative Shannon's entropy.
  - Of course $I = I_1 + I_2 - \Delta I$

Multipartite systems

- Quantum mutual information (Total correlation) between the two subsystems $\hat{\rho}_1$ and $\hat{\rho}_2$ of the joint state $\hat{\rho}_{12}$

$$I(\hat{\rho}_1 : \hat{\rho}_2; \hat{\rho}_{12}) = S(\hat{\rho}_1) + S(\hat{\rho}_2) - S(\hat{\rho}_{12})$$

where $S = -Tr\hat{\rho}\log_2\hat{\rho}$ von Neumann entropy

- Quantum relative entropy between $\hat{\sigma}$ and $\hat{\rho}$

$$S(\hat{\sigma}||\hat{\rho}) = Tr[\hat{\sigma}(\log_2\hat{\sigma} - \log_2\hat{\rho})]$$

- Quantum mutual information is a distance of $\hat{\rho}_{12}$ to the closest uncorrelated $\hat{\rho}_1 \otimes \hat{\rho}_2$

$$I(\hat{\rho}_1 : \hat{\rho}_2; \hat{\rho}_{12}) = S(\hat{\rho}_{12}||\hat{\rho}_1 \otimes \hat{\rho}_2)$$

- Multipartite quantum mutual information in $\hat{\rho}$ (generalization)

$$I(\hat{\rho}_1 : \hat{\rho}_2 : \ldots : \hat{\rho}_n; \hat{\rho}) = S(\hat{\rho}||\hat{\rho}_1 \otimes \hat{\rho}_2 \otimes \ldots \otimes \hat{\rho}_n) = \sum_i S(\hat{\rho}_i) - S(\hat{\rho})$$

e.g. $S(\hat{\rho}||\hat{\rho}_{MF}) = \ln Z_{MF} - \ln Z + \beta\langle H_{MF} - H\rangle_H \geqslant 0.$

Entanglement for multipartite system

Relative entanglement

$$E(\hat{\rho}) = min_{\hat{\sigma}\in\{\text{separable}\}} S(\hat{\rho}||\hat{\sigma})$$

the relative entanglement is a distance between $\hat{\rho}$ and the closest classically correlated state

$$E(\hat{\rho}) \leqslant I(\hat{\rho})$$

If we take $\hat{\sigma} = \hat{\rho}_{MF}$ (???)

$$E(\hat{\rho}) = \ln Z_{MF} - \ln Z + \beta\langle H_{MF} - H\rangle_H$$

$$\chi = \frac{\partial^2 \ln Z}{\partial B^2}, \qquad \chi_{sep} - \chi = \frac{\partial^2 E(\hat{\rho})}{\partial B^2} + \beta \frac{\partial^2 \langle H_{MF} - H\rangle_H}{\partial B^2}$$

Th. general bound for multipartite entanglement (Vedral 2003)

$$E(\hat{\rho}) \leqslant \ln Z_{MF} - \ln Z + \beta\langle H_{MF} - H\rangle_H .$$

Entanglement and the III law

Nernst theorem says that $S(T) \to S_0 = const$, or equivalently $C_V(T) = T(\partial S(T)/\partial T)_V \to 0$ when $T \to 0$.

Th. Wiesniak *et al.* (2005): Only if entanglement develops at low temperatures the Nernst theorem is satisfied.

because: separable states give bound for the ground state energy $U(T=0) \geqslant E_B$ and hence for all separable states

$$C = \frac{\partial U(T)}{\partial T} = \gamma \frac{U(T) - E_0}{T} \geqslant \gamma' \frac{E_B - E_0}{T^{1(2)}}$$

1 for gapless, 2 for gapped systems.
Only when $E_B = E_0$, $C(T) \to 0$
In general $C(T) \to \infty$ for all separable states.

Indistinguishable particles

- if wave function overlap very small then all effects due to statistics can be neglected; well separated particles in space
- if wave function overlap important
  - Pauli (statistical) entanglement (due to anty/symmetrization); probably not of use in teleportation etc.
  - entanglement between different determinants/permanents
  - mode entanglement (Fock space entanglement); the entanglement is basis (operator, observer) dependent, e.g.

$$H = \sum_{ij} t_{ij} a_i^\dagger a_j = \sum_{\mathbf{k}} \epsilon_{\mathbf{k}} a_{\mathbf{k}}^\dagger a_{\mathbf{k}}$$

Indistinguishable particles

Example:

- single particle at A or B

$$|0,1\rangle + |1,0\rangle \quad \text{in mode (Fock) reprezentation}$$

$$\psi_A(x) + \psi_B(x) \quad \text{in coordinate reprezentation}$$

- two particles, one at A, one at B

$$|1,1\rangle \quad \text{in mode (Fock) reprezentation}$$

$$\psi_A(x)\psi_B(y) \pm \psi_A(y)\psi_B(x) \quad \text{in coordinate reprezentation}$$

- two particles, in different modes (as in (i)) superposition

$$(|0,1\rangle + |1,0\rangle)(|0,1\rangle + |1,0\rangle) = |00,11\rangle + |01,10\rangle + |10,01\rangle + |11,00\rangle$$

$$(\psi_{A_1}(x) + \psi_{B_1}(x))(\psi_{A_2}(y) + \psi_{B_2}(y)) \pm (x \leftrightarrow y)$$

One needs more refined measure of entanglement then simply von Neumann entropy $S(\hat{\rho}_A)$ even for pure states of indistinguishable particles.

Summary

- Entanglement is a quantum correlation in quantum many body system
- Entanglement does not depend on particular physical representation
- Correlation without correlata
- Entanglement is a resource for certain tasks
- Entanglement can be quantified and measured