Dynamical mean-field theory for correlated bosons and fermions on a lattice in condensed and normal phases

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October 17th, 2008
Main results

- New comprehensive dynamical mean-field theory for correlated, lattice bosons in normal and condensate phases, exact in $d \to \infty$

- Correlation might enhance BEC fraction and transition temperature
- DMFT for bose-fermi mixtures
- Real-space formulation of DMFT
Collaboration

Dieter Vollhardt - Augsburg University

*Correlated bosons on a lattice: Dynamical mean-field theory for Bose-Einstein condensed and normal phases*

FERMIONS
Correlated lattice fermions

\[ H = - \sum_{ij\sigma} t_{ij} c_{i\sigma}^{\dagger} c_{j\sigma} + U \sum_i n_{i\uparrow} n_{i\downarrow} \]

fermionic Hubbard model, 1963

Local Hubbard physics

\[ |i, 0\rangle \rightarrow |i, \uparrow\rangle \rightarrow |i, 2\rangle \rightarrow |i, \downarrow\rangle \]
The Holy Grail for correlated electrons (fermions)

Fact: Hubbard model is not solved for arbitrary cases

Find the best comprehensive approximation

- valid for all values of parameters $t, U, n = N_e/N_L, T$, all thermodynamic phases
- thermodynamically consistent
- conserving
- possessing a small expansion (control) parameter and exact in some limit
- flexible to be applied to different systems and material specific calculations
Fermions in large dimensions

Large dimensional limit is not unique

No scaling at all:

\[ t_{ij} = t_{ij}, \quad U = U, \quad \text{etc.} \]

\[ \frac{1}{N_L} E_{\text{kin}} = \frac{1}{N_L} \sum_{ij\sigma} t_{ij} \langle c_{i\sigma}^\dagger c_{j\sigma} \rangle = \infty ! \]

Overscaling fermions

\[ t_{ij} = \frac{t_{ij}^*}{d ||R_i - R_j||}, \quad U = U, \quad \text{etc.} \]

\[ \frac{1}{N_L} E_{\text{kin}} = \frac{1}{N_L} \sum_{ij\sigma} t_{ij} \langle c_{i\sigma}^\dagger c_{j\sigma} \rangle = 0 ! \]
Fermions in large dimensions (coordination)

Non-trivial (asymptotic) theory is well defined such that the energy density is generically finite and non-zero

\[
\frac{1}{N_L} E_{\text{kin}} = \frac{1}{N_L} \sum_{i,j,\sigma} t_{ij} \langle c_{i\sigma}^\dagger c_{j\sigma} \rangle = \frac{1}{N_L} \sum_{i,\sigma} \sum_{j(\cdot)} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi i} G_{ij\sigma}(\omega) \sim O(1)
\]

Fact, since \( G_{ij} \) is probability amplitude for hopping,

\[
G_{ij} \sim O(d^{-\frac{||R_i-R_j||}{2}})
\]

with rescaling

\[
t_{ij} \rightarrow \frac{t_{ij}^*}{\sqrt{d||R_i-R_j||}}
\]

sum \( \sum_{j(\cdot)} \) is compensated and energy is finite (Metzner, Vollhardt, 1989)
Comprehensive mean-field theory for fermions

\[ H = H^{\text{hopping}} + H^{\text{interaction}} \]

- comprehensive (all input parameters, temperatures, all phases, ...)
- thermodynamically consistent and conserving
- provides exact solutions in certain non-trivial limit (large \( d \))

\[ \langle H \rangle, \quad \langle H^{\text{hopping}} \rangle, \quad \langle H^{\text{interaction}}_{\text{loc}} \rangle \]

are finite and generically non-zero, and

\[ \langle [H^{\text{hopping}}, H^{\text{interaction}}_{\text{loc}}] \rangle \neq 0 \]

to describe non-trivial competition
Non-comprehensive mean-field theory for fermions

- Distance independent hopping (van Dongen, Vollhardt 92)

\[ H = t \sum_{ij\sigma} c_{i\sigma}^{\dagger} c_{j\sigma} + U \sum_{i} n_{i\uparrow} n_{i\downarrow} \]

- Distance independent interaction (Spalek, Wojcik 88, KB 92-94, Baskaran 91, Kohmoto 95, Gebhard 97)

\[ H = \sum_{ij\sigma} t_{ij} c_{i\sigma}^{\dagger} c_{j\sigma} + U \sum_{ij} n_{i\uparrow} n_{j\downarrow}, \quad H = \sum_{k\sigma} \epsilon_{k} n_{k\sigma} + U \sum_{k} n_{k\uparrow} n_{k\downarrow} \]

In both models a non-trivial competition is suppressed

\[ \langle [H_{\text{hopping}}, H_{\text{interaction}}] \rangle = 0 \]

although \( \langle H \rangle, \quad \langle H_{\text{hopping}} \rangle, \quad \langle H_{\text{interaction}} \rangle \neq 0 \)
\( d \to \infty \) limit – Feynman diagrams simplification

One proves, term by term, that skeleton expansion for the self-energy \( \Sigma_{ij}[G] \) has only local contributions

\[
\Sigma_{ij\sigma}(\omega_n) \to_{d\to\infty} \Sigma_{i\sigma}(\omega_n) \delta_{ij}
\]

Fourier transform is \( k \)-independent

\[
\Sigma_{\sigma}(k,\omega_n) \to_{d\to\infty} \Sigma_{\sigma}(\omega_n)
\]

DMFT is an exact theory in infinite dimension (coordination number) and a small control parameter is \( 1/d \) (1/\( z \))

(Metzner, Vollhardt, 1989; Muller-Hartmann, 1989; Georges, Kotliar, 1990'; Janis, Vollhardt 1990', ...)
DMFT for lattice fermions

Replace (map) full many-body lattice problem by a single-site coupled to dynamical reservoir and solve such problem self-consistently

All local dynamical correlations included exactly

Space correlations neglected - mean-field approximation
DMFT - equations full glory

Local Green function

\[ G_\sigma(\tau) = -\langle T_\tau c_\sigma(\tau) c_\sigma^*(0) \rangle_{S_{loc}} \]

where

\[ S_{loc} = -\sum_\sigma \int d\tau d\tau' c_\sigma^*(\tau) G_\sigma^{-1}(\tau - \tau') c_\sigma(\tau') + U \int d\tau n_\uparrow(\tau) n_\downarrow(\tau) \]

Weiss (mean-field) function and self-energy

\[ G_\sigma^{-1}(\omega_n) = G_\sigma^{-1}(\omega_n) + \Sigma_\sigma(\omega_n) \]

Local Green function and lattice system self-consistency

\[ G_\sigma(i\omega_n) = \sum_\mathbf{k} G_\sigma(\mathbf{k}, \omega_n) = \sum_\mathbf{k} \frac{1}{i\omega_n + \mu - \epsilon_\mathbf{k} - \Sigma_\sigma(\omega_n)} = G_\sigma^0(i\omega + \mu - \Sigma_\sigma(\omega_n)) \]
DMFT – flexibility; LDA+DMFT

Multi-band systems (Anisimov et al. 97; ... Nekrasov et al. 00, ...)

\[ H = H_{\text{LDA}} + H_{\text{int}} - H_{\text{LDA}}^U = H_{\text{LDA}}^0 + H_{\text{int}} \]

direct and exchange interaction

\[ H_{\text{int}} = \frac{1}{2} \sum_{i=i_d, l=l_d} \sum_{m\sigma, m'\sigma'} U_{mm'}^\sigma\sigma' n_{ilm\sigma} n_{ilm'\sigma'} \]

\[ -\frac{1}{2} \sum_{i=i_d, l=l_d} \sum_{m\sigma, m'} J_{mm'} c_{ilm\sigma}^\dagger c_{ilm'\sigma}^\dagger - c_{ilm'\sigma} c_{ilm-\sigma} \]

kinetic part, determined from DFT-LDA calculation (material specific)

\[ H_{\text{LDA}}^0 = \sum_{ilm, jl' m', \sigma} t_{ilm, jl' m'}^0 c_{ilm\sigma}^\dagger c_{jl' m' \sigma} \]

LDA+DMFT - state of the art for realistic approach to correlated electron systems
DMFT scheme

$S_{loc}$ - local interactions $U$ or $J$ from a model TB or a microscopic LDA Hamiltonian

$\hat{G} = -\langle T\hat{C}(\tau)\hat{C}^*(0) \rangle_{S_{loc}}$

$\hat{\Sigma} = \hat{G}^{-1} - \hat{G}^{-1}$

$\hat{G}^{-1} = \hat{G}^{-1} + \hat{\Sigma}$

$\hat{G} = \sum[(\omega + \mu)\hat{1} - \hat{H}^0 - \hat{\Sigma}]^{-1}$

$\hat{H}^0$ is a model TB or a microscopic LDA Hamiltonian

D. Vollhardt

W. Metzner

G. Kotliar

A. Georges
BOSONS
Short-history: Bose-Einstein distribution

- M. Planck 1900 - introduces distinguishable quanta of energy and gets BE function

\[
\bar{e}(\omega) = \sum_{n=0}^{\infty} \hbar \omega n e^{\frac{\hbar \omega}{k_B T}} = \frac{\hbar \omega}{e^{\frac{\hbar \omega}{k_B T}} - 1}.
\]

- L. Natanson 1911 - introduces concept of indistinguishable photon quanta and by combinatorial way gets BE distribution function

\[
n_\epsilon = \frac{1}{e^{\frac{\epsilon}{k_B T}} - 1},
\]

in *On the statistical theory of Radiation*, Bulletin de l’Académie des Sciences de Cracovie (A), 134 (1911); German translation in Phys. Z. 12, 659 (1911).

Władysław Natanson, 1864-1937
Short-history: Bose-Einstein condensation (BEC)

A. Einstein 1925 - considers conserved bosons ($\mu$ defined) and observes that

$$N = \int_0^{\infty} d\epsilon \frac{N_0(\epsilon)}{e^{\frac{\epsilon - \mu}{k_BT}} - 1}$$

is *only* correct for $N \leq N_c = \zeta(3/2)(2\pi\hbar^2/mk_BT)^{3/2}$ in $d = 3$, cf. Berl. Ber. 22, 261 (1924), ibid. 23, 3 and 18 (1925).

$$\frac{2\pi\hbar}{\sqrt{2mk_BT}} = \lambda_d B > a_0 = \left(\frac{N}{V}\right)^{-\frac{1}{d}}$$

Bose-Einstein Condensation: for conserved bosons above $N_c$ (below $T_c$) the lowest energy state is occupied by macroscopically large number of bosons. Then $\mu = 0$ and the state $\epsilon_{k=0}$ must be treated separately

$$N = N_c + \int_0^{\infty} d\epsilon \frac{N_0(\epsilon)}{e^{\frac{\epsilon}{k_BT}} - 1}.$$
Short-history: general BEC as ODLRO

O. Penrose 1951, O. Penrose and L. Onsager 1956
Off-Diagonal Long Range Order (ODLRO) as a general definition of BEC for interacting bosons in any ensemble, external potential, etc.

One-particle reduced density matrix

\[ \rho(r, r'; t) \equiv N \sum_s p_s \int dr_2...dr_N \Psi_s^\dagger(rr_2...r_N; t) \Psi_s(r'r_2...r_N; t) = \langle \psi^\dagger(rt)\psi(r't) \rangle \]

Spectral decomposition (diagonalization)

\[ \rho(r, r'; t) = \sum_\alpha n_\alpha(t) \chi_\alpha^\dagger(rt)\chi_\alpha(r't) \]

BEC occurs when there exists one-particle state(s) \( \alpha = 0 \) for which \( n_0 = N_c \sim O(N) \)
Short history: True experiments with BEC

Magneto-optical traps with cold alkaline atoms with Bose statistics ($^7$Li, $^{23}$Na, $^{41}$K, $^{52}$Cr, $^{85}$Rb, $^{87}$Rb, $^{133}$Cs and $^{174}$Yb)

Short history: Superfluid-Mott transition

Optical lattices with cold atoms

Superfluid-Mott insulator transition, Greiner, Mandel, Esslinger, Hänsch, Bloch, 2002
**Correlated bosons on optical lattices**

bosonic Hubbard model

\[ H = \sum_{ij} t_{ij} b_i^\dagger b_j + \frac{U}{2} \sum_i n_i(n_i - 1) \]

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**local (on-site) correlations in time**

\[ E_{\text{int}} = U \]

\[ \begin{array}{cccc}
U & 3U & 6U & 3U \\
\end{array} \]

\[ |i, 2\rangle \rightarrow |i, 3\rangle \rightarrow |i, 4\rangle \rightarrow |i, 3\rangle \]

integer occupation of single site changes in time

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Gersch, Knollman, 1963
Fisher et al., 1989
Scalettar, Kampf, et al., 1995
Jacksch, 1998
**Standard approximations**

- Bose-Einstein condensation treated by Bogoliubov method $b_i = \langle b_i \rangle + \tilde{b}_i$ where $\langle b_i \rangle \equiv \phi_i \in C$ classical variable (Bogoliubov 1947)

- Weak coupling - mean-field (expansion) in $U$, valid for small $U$, average on-site density, local correlations in time neglected (Ooste, Stoof, et al., 2000)

- Strong coupling - mean-field (expansion) in $t$, valid for small $t$ (Freericks, Monien, 1994; Kampf, Scalettar, 1995)

Bose-Einstein condensate – Mott insulator transition

$U \sim t$

intermediate coupling problem

**Comprehensive mean-field theory needed**

Like DMFT for fermions: exact and non-trivial in $d \to \infty$ limit
Quantum lattice bosons in $d \to \infty$ limit

W. Metzner and D. Vollhardt 1989 - rescaling of hopping amplitudes for fermions

\[ t_{ij} = \frac{t^*_{ij}}{||R_i - R_j||} \] for NN $i, j$ \[ t = \frac{\sqrt{2}}{\sqrt{2d}} \]

Not sufficient for bosons because of BEC:

One-particle density matrix at $||R_i - R_j|| \to \infty$

\[ \rho_{ij} = \langle b_i^\dagger b_j \rangle = \frac{N_c}{N_L} \underbrace{1}_{\text{BEC part}} + \frac{1}{N_L} \sum_{k \neq 0} n_k e^{ik(R_i-R_j)} \underbrace{\text{normal part}}_{||R_i-R_j|| \to \infty} \]

\[ \frac{N_c}{N_L} = n_c \]

- BEC part – constant
- normal part – vanishes

The two contributions to the density matrix behave differently
Quantum lattice bosons in $d \rightarrow \infty$ limit

- No scaling:
  \[
  \frac{1}{N_L} E_{kin} = \infty
  \]

- Fractional scaling:
  \[
  \frac{1}{N_L} E_{kin} = \infty
  \]
  in the BEC phase

- Integer scaling:
  \[
  \frac{1}{N_L} E_{kin} = 0
  \]
  in the normal phase

No way to construct comprehensive mean-field theory in the bare Hamiltonian operator formalism
BEC and normal bosons on the lattice in $d \to \infty$ limit

1. Rescaling is made inside a thermodynamical potential (action, Lagrangian) but not at the level of the Hamiltonian operator

   - normal parts: $t_{ij} = \frac{t_{ij}^*}{||(R_i - R_j)||}$ - fractional rescaling
   - BEC parts: $t_{ij} = \frac{t_{ij}^*}{(2d)||R_i - R_j||}$ - integer rescaling

2. Limit $d \to \infty$ taken afterwards in this effective potential

Only this procedure gives consistent derivation of B-DMFT equations as exact ones in $d \to \infty$ limit for boson models with local interactions
Bosonic-Dynamical Mean-Field Theory (B-DMFT)

- Exact mapping of the lattice bosons in infinite dimension onto a single site
- Single site coupled to two reservoirs: normal bosons and bosons in the condensate
- Reservoirs properties are determined self-consistently, local correlations kept
B-DMFT application to bosonic Hubbard model

(i) Lattice self-consistency equation (exact in \( d \to \infty \))

\[
\hat{G}(i\omega_n) = \int d\epsilon N_0(\epsilon) \left[ \begin{pmatrix} i\omega_n + \mu - \epsilon & 0 \\ 0 & -i\omega_n + \mu - \epsilon \end{pmatrix}^{-1} - \hat{\Sigma}(i\omega_n) \right]^{-1}
\]

(ii) Local impurity

\[
\hat{G}(\tau) = \int D[b^*, b] \bar{b}(\tau) b^*(0) e^{-S_{loc}}
\]

\[
S_{loc} = -\int_0^\beta \int_0^\beta d\tau d\tau' \bar{b}(\tau) \hat{G}^{-1}(\tau - \tau') b(\tau) + \kappa \int_0^\beta d\tau \bar{\phi}^\dagger(\tau) \bar{b}(\tau) + \frac{U}{2} \int_0^\beta n(\tau)(n(\tau) - 1)
\]

\[
\hat{G}^{-1}(i\omega_n) = \hat{G}^{-1}(i\omega_n) + \hat{\Sigma}(i\omega_n) = \begin{pmatrix} i\omega_n + \mu & 0 \\ 0 & -i\omega_n + \mu \end{pmatrix} - \hat{\Delta}(i\omega_n)
\]

(iii) Condensate wave function

\[
\bar{\phi}(\tau) = \int D[b^*, b] \bar{b}(\tau) e^{-S_{loc}}
\]
B-DMFT in well-known limits

Bosonic Hubbard Model \((t_{ij}, U)\)

\[ t_{ij} = \frac{\tilde{t}_{ij}}{(Z^{R})^{s}} \]

\( s = 1/2 \) for normal bosons
\( s = 1 \) for condensed bosons

\[ d \rightarrow \infty \]
\[ N_{L} \rightarrow \infty \]

Beliaev-Popov approximation

Hartree-Fock-Bogoliubov approximation

Keep only diagrams of 1. order in \( U \)

Normal diagrams = 0

Bogoliubov mean-field theory

B-DMFT

\( \Lambda = 0 \) (Immobile normal bosons)

Boson mean-field theory of Fisher et al.

Free bosons

Immobile bosons

\[ U = 0 \]

\[ t_{ij} = 0 \]
B-DMFT Generalized Gross-Pitaevskii equation

Condensate wave function

\[ \Phi(\tau) = \langle b(\tau) \rangle_{S_{loc}} \leftrightarrow \langle \delta S_{loc}/\delta b^* \rangle_{S_{loc}} = 0 \]

Approximation: \( b(\tau) \rightarrow \langle b(\tau) \rangle_{S_{loc}} \equiv \Phi(\tau) \)

\[ \frac{\partial \Phi(\tau)}{\partial \tau} = \left[ \mu - \kappa - U|\Phi(\tau)|^2 \right] \Phi(\tau) + \int_{0}^{\beta} d\tau' \left[ \Delta^{11}(\tau - \tau')\Phi(\tau') + \Delta^{12}(\tau - \tau')\Phi^*(\tau') \right] \]

Generalized Gross-Pitaevski equation includes retardation effects due to normal bosons
**B-DMFT application to bosonic Falicov-Kimball model**

Binary mixture of itinerant (b) and localized (f) bosons on the lattice

\[
H = \sum_{ij} t_{ij} b_i^\dagger b_j + \epsilon_f \sum_i f_i^\dagger f_i + U_{bf} \sum_i n_{bi} n_{fi} + U_{ff} \sum_i n_{fi} n_{fi}
\]

Local conservation law \([n_{fi}, H] = 0\) hence \(n_{fi} = 0, 1, 2, \ldots\) classical variable

B-DMFT: local action Gaussian and **analytically integrable**
Enhancement of $T_{BEC}$ due to interaction

Hard-core f-bosons $U_{ff} = \infty$; $n_f = 0, 1$; $0 \leq \bar{n}_f \leq 1$; $d = 3$ - SC lattice

\[ A_b(\omega) = -\text{Im}G_b(\omega)/\pi \]

\[ \bar{n}_b = \bar{n}^{BEC}_b + \int d\omega \frac{A_b(\omega+\mu_b)}{e^{\omega/T}-1} \]

Normal part decreases when $U$ increases for constant $\mu_b$ and $T$
Exact limit: enhancement of $T_{BEC}$ due to interaction

Hard-core f-bosons $U_{ff} = \infty$; $n_f = 0, 1$; $0 \leq \bar{n}_f \leq 1$; $d = \infty$ - Bethe lattice

\begin{align*}
A_b(\omega) &= -\text{Im}G_b(\omega)/\pi \\
\bar{n}_b &= \bar{n}_b^{BEC} + \int d\omega \frac{A_b(\omega+\mu_b)}{e^{\omega/T}-1} \\
\end{align*}

Normal part decreases when $U$ increases for constant $\mu_b$ and $T$
BOSONS and FERMIONS
Bose-Fermi mixtures ($^{87}\text{Rb} -^{40}\text{K}$) on a lattice with a trap

$$H = \sum_{ij} t_{ij}^b b_i^\dagger b_j + \sum_i \epsilon_i^b n_i^b + \frac{U_b}{2} \sum_i n_i^b (n_i^b - 1) + \sum_{ij} t_{ij}^f f_i^\dagger f_j + \sum_i \epsilon_i^f n_i^f + U_{bf} \sum_i n_i^b n_i^f$$
DMFT for bose-fermi mixtures

BF-DMFT equations:

\[
S_{i_0}^{b} = \int_{0}^{\beta} d\tau b_{i_0}^\dagger(\tau) \left( \partial_\tau \sigma_3 - (\mu_b - \epsilon_{i_0}^{b})1 \right) b_{i_0}(\tau) + \int_{0}^{\beta} d\tau \int_{0}^{\beta} d\tau' b_{i_0}^\dagger(\tau) \Delta_{i_0}^{b}(\tau - \tau') b_{i_0}(\tau') \\
+ U_b \int_{0}^{\beta} n_{i_0}^{b}(\tau)(n_{i_0}^{b}(\tau) - 1) + \int_{0}^{\beta} d\tau \sum_{j \neq i_0} t_{i_0 j}^{b} b_{i_0}^\dagger(\tau) \Phi_{j}(\tau)
\]

\[
S_{i_0}^{f} = \int_{0}^{\beta} d\tau f_{i_0}^*(\tau) \left( \partial_\tau - \mu_f + \epsilon_{i_0}^{f} \right) f_{i_0}(\tau) + \int_{0}^{\beta} d\tau \int_{0}^{\beta} d\tau' f_{i_0}^*(\tau) \Delta_{i_0}^{f}(\tau - \tau') f_{i_0}(\tau')
\]

\[
S_{i_0}^{bf} = U_{bf} \int_{0}^{\beta} d\tau n_{i_0}^{b}(\tau)n_{i_0}^{f}(\tau)
\]
Lattice self-consistency (Dyson) equations

\[ G^b_{ij}(i\nu_n) = \left[ (i\nu_n\sigma_3 + \mu_b 1 - \Sigma_i^b (i\nu_n))\delta_{ij} - t^b_{ij} 1 \right]^{-1} \]

\[ G^f_{ij}(i\omega_n) = \left[ (i\omega_n + \mu_f - \Sigma_i^f (i\omega_n))\delta_{ij} - t^f_{ij} \right]^{-1} \]
Integrating out fermions

\[ Z_{i0}^{\text{loc}} = \int D[b] e^{-S_{i0}^{b}[b]} + \ln \det \left[ M_{i0}^{b} \right] \]

\[ [M_{i0}^{b}]_{nm} \equiv \left[ (\partial_{\tau} - \mu_f + \epsilon_{i0}^f + U_{bf} n_{i0}^{b}(\tau)) \delta_{\tau,\tau'} + \Delta_{i0}^f (\tau - \tau') \right]_{nm} \]

\[
= \left[ -i\omega_n - \mu_f + \epsilon_{i0}^f + \Delta_{i0}^f (\omega_n) \right] \delta_{nm} + \frac{U_{bf}}{\sqrt{\beta}} n_{i0}^{b}(\omega_n - \omega_m)\]
Effective interaction between bosons

$$\ln \text{Det}[M^b] = \text{Tr} \ln[M^b] = \text{Tr} \ln[-(G^f)^{-1} + M_1^b] = \text{Tr} \ln[-(G^f)^{-1}] - \sum_{m=1}^{\infty} \frac{1}{m} \text{Tr}[G^f M_1^b]^m$$

$$G^f_{i0}(\omega_n) = \frac{1}{i\omega_n + \mu_f - \epsilon^f_{i0} - \Delta^f_{i0}(\omega_n)}$$

Effective bosonic action

$$\tilde{S}^b_{i0} \approx S^b_{i0} + \frac{U_{bf}}{\sqrt{\beta}} \sum_n G^f_{i0}(\omega_n)n^b_{i0}(\nu_m = 0) - \frac{U_{bf}^2}{2} \sum_n n^b_{i0}(\nu_n)\pi^f_{i0}(\nu_n)n^b_{i0}(-\nu_n)$$

$$\pi^f_{i0}(\nu_n) \equiv -\frac{1}{\beta} \sum_m G^f_{i0}(\omega_m)G^f_{i0}(\omega_m + \nu_n)$$

Boson-Boson interaction

$$U^\text{eff}_b = U_b - U_{bf}^2 N^f_{i0}(\mu)$$

System unstable when $$U_b = U_{bf}^2 N^f_{i0}(\mu)$$.
Summary and Outlook

- Formulated Bosonic Dynamical Mean-Field Theory (B-DMFT)
  - comprehensive mean-field theory
  - conserving and thermodynamically consistent
  - exact in $d \to \infty$ limit due to new rescaling

- B-DMFT equations for bosonic Hubbard model

- B-DMFT solution for bosonic Falicov-Kimball model
  - Enhancement of $T_{BEC}$ due to correlations
  - Mixture of $^{87}$Rb (f-bosons) and $^7$Li (b-bosons) may have larger $T_{BEC}$ on optical lattices

- Spinor bosons, bose-fermi mixture within B-DMFT or density like LRO easy to include within B-DMFT

- Bosonic impurity solver wanted!