Verschränkung

Krzysztof Byczuk

Institute of Physics, Augsburg University

November 23th, 2005

What you would like to know about

entanglement

but you were afraid to ask

Main goal:

- Entanglement is a quantum correlation in quantum many body system
- Entanglement does not depend on particular physical representation
- Entanglement is a resource like energy
- Entanglement can be quantified and measured

Plan of the talk:

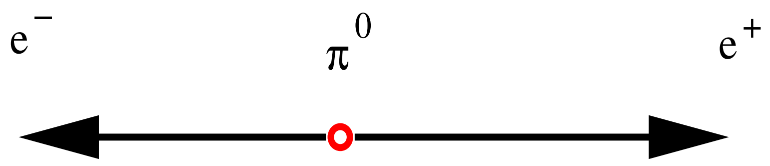
1. EPR and Bell story (introduction of entanglement)
2. How to use entanglement
   - no cloning theorem, quantum teleportation
3. How to characterize entanglement
   - pure vs. mixed states entanglement
   - measures of entanglement
4. How to quantify correlations in bulk systems
5. Conclusions and outlook: *correlations without correlata?*
6. Appendix
   - How to entangle photons, electrons, ...experimentally
   - Beating diffraction Rayleigh limit in classical wave optics

EPR theorem

Einstein, Podolsky, Rosen (1935)

$$\mathcal{H} = \mathcal{H}_+ \otimes \mathcal{H}_-$$

$$|\Psi\rangle = [|\uparrow\rangle_- \otimes |\downarrow\rangle_+ - |\downarrow\rangle_- \otimes |\uparrow\rangle_+]/\sqrt{2}$$



$\pi^0 \longrightarrow e^+ + e^-$

$S_{\mathrm{tot}} = 0$ and $S^z = 0$ - singlet state (Bohm 1954)

Orthodox (Copenhagen) view:

neither particle had either spin up or spin down until the act of measurement intervented: your measurment of e^- collapsed the wave function, and instanteneusly "produced" the spin of e^+ 20 ly. far away

EPR - $|\Psi\rangle$ does not provide a complete description of physical reality with locality principle

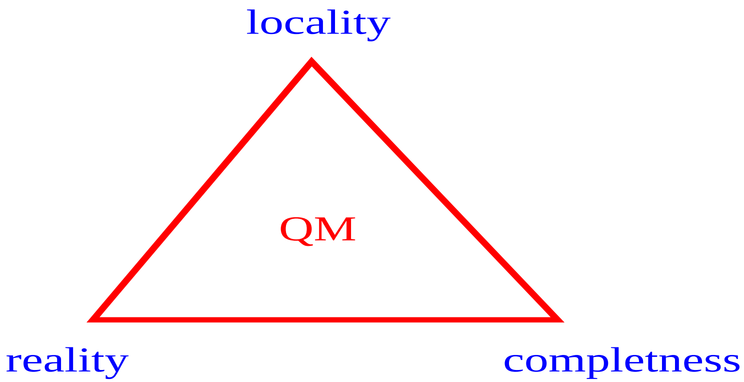
spooky action at a distance, hidden variable, ghost field, ..., to keep locallity

EPR theorem today

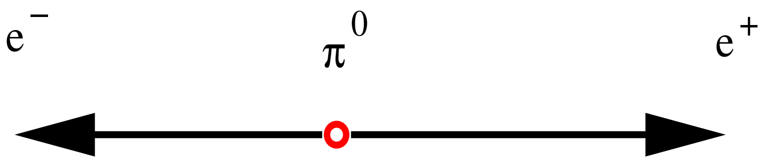
$$\mathcal{H} = \mathcal{H}_+ \otimes \mathcal{H}_-$$

$$|\Psi_{\rm EPR}\rangle = [|\uparrow\rangle_- \otimes |\downarrow\rangle_+ - |\downarrow\rangle_- \otimes |\uparrow\rangle_+]/\sqrt{2}$$

Verschränkung - entanglement (Schrödinger 1935)



QM is nonlocal



correlations over distance

results of independent measurements will be correlated

no superluminal transfer of information, energy, etc.

Bell theorem

J. Bell (1964)

any local hidden variable theory is incompatible with quantum mechanics

- has found inequalities for correlation functions that are violated in QM (G. Boole (1862))
- first approach to quantify entanglement
- A. Aspect *et al.* (1982) first experiment
- now many others, hopefully without loopholes

NATURE ITSELF IS FUNDAMENTALLY NONLOCAL, EXPRESSED IN A SUBTLE CORRELATIONS BETWEEN TWO LISTS OF OTHERWISE RANDOM DATA

Bipartite pure entanglement

Let $\{|i\rangle_A \otimes |j\rangle_B\} \in \mathcal{H} = \mathcal{H}_\mathcal{A} \otimes \mathcal{H}_\mathcal{B}$ and AB distinguishable.

Any state

$$|\Psi\rangle = \sum_{ij} \gamma_{ij} |i\rangle_A \otimes |j\rangle_B$$

that cannot be represented as a product state is called an entangled state.

- Entanglement is a quantum correlation which does not have a classical counterpart
- any entangled state cannot be prepared from a product state by local operations and classical communications (LOCC)

Bell states

- classical two level system (0 or 1) codes one bit of information
- in QM two level system can be both 0 and 1 (spin, polarization, vortex, energy structure)
- it was proposed to call it quantum bit or qbit (read: *qiubit*) in general - Schumacher (1995)

Bell states - maximally entangled states of two qbits

$$|\Psi^-\rangle = \frac{1}{\sqrt{2}}\left[|01\rangle - |10\rangle\right]$$

$$|\Psi^+\rangle = \frac{1}{\sqrt{2}}\left[|01\rangle + |10\rangle\right]$$

$$|\Phi^-\rangle = \frac{1}{\sqrt{2}}\left[|00\rangle - |11\rangle\right]$$

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}\left[|00\rangle + |11\rangle\right]$$

No cloning theorem

Wootters, Zurek, and Dieks (1982)

the unknown quantum state cannot be copied

a.a.
let $\hat{U}_{\text{clon}}$ exists, i.e. $\hat{U}_{\text{clon}}|\phi\rangle|0\rangle = |\phi\rangle|\phi\rangle$
and $|\phi\rangle = \alpha|0\rangle + \beta|1\rangle$ unknown

$$\hat{U}_{\text{clon}}|\phi\rangle|0\rangle = |\phi\rangle|\phi\rangle = (\alpha|0\rangle+\beta|1\rangle)(\alpha|0\rangle+\beta|1\rangle) = \alpha^2|00\rangle+\alpha\beta(|01\rangle+|10\rangle)+\beta^2|11\rangle$$

but

$$\hat{U}_{\text{clon}}|\phi\rangle|0\rangle = \hat{U}_{\text{clon}}(\alpha|0\rangle+\beta|1\rangle)|0\rangle = \hat{U}_{\text{clon}}(\alpha|00\rangle+\beta|10\rangle) = \alpha|00\rangle+\beta|11\rangle$$

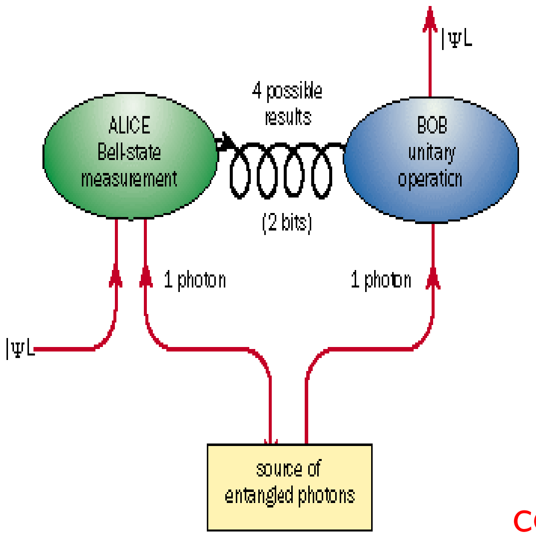
contradiction!

Quantum teleportation

Bennett *et al.* (1993), photons (1998-2005), atoms (2004)

Alice and Bob share one entangled state, e.g. $|\Phi^+\rangle$.
Alice wants to send to Bob all necessary information about the unknown quantum state $|\Phi\rangle = a|0\rangle + b|1\rangle$ she has got such that Bob could recreate this state using a particle he has at hand. This is a task of quantum teleportation. The state at Alice will be destroyed.
What about the entangled state they share?

$$|\Phi\rangle|\Phi^+\rangle \sim [|\Phi^+\rangle(a|0\rangle + b|1\rangle) + |\Phi^-\rangle(a|0\rangle - b|1\rangle) + |\Psi^+\rangle(a|1\rangle + b|0\rangle) + |\Psi^-\rangle(a|1\rangle - b|0\rangle)]$$



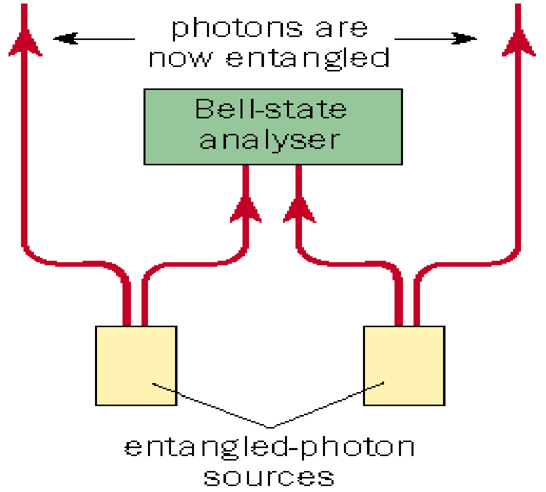
A: performs projective measurement on her 2 qbits - LO

A: call Bob and tells her result (one of 4) - CC

B: depending on A info performs 1 or σ_x or/and σ_z - LO

cost: one Bell state is eatten up

Entanglement swapping



Qbit at Alice can be in entangled state with another qbit

Quantum teleporting one qbit she can exchange entanglement

Entanglement is swoped between different particles

Again it costs one Bell state

Mixed state

- density operator $\hat{\rho} = \sum_n p_n |\Psi_n\rangle\langle\Psi_n|$ describes a system coupled to another system to which we do not have an access
- pure state - maximal knowledge $\hat{\rho}^2 = \hat{\rho}$
- mixed state - statistical knowledge, mixture of different pure states can lead to the same density operator and thereby the same mixed state
- states from different ensembles having the same density operator are experimentally indistinguishable
- when pure system has entangled subsystems then each subsystem is in a mixed state, e.g.

$$|\Psi\rangle = \alpha|00\rangle + \beta|11\rangle$$

reduced density operator

$$\hat{\rho}_A = Tr_B\hat{\rho} = Tr_B|\Psi\rangle\langle\Psi| = |\alpha|^2|0\rangle\langle 0| + |\beta|^2|1\rangle\langle 1|$$

Entanglement in mixed state

A mixed state is not entangled if there exists a convex decomposition into pure product state of its density operator, i.e.

$$\hat{\rho} = \sum_n p_n |\Psi_n\rangle\langle\Psi_n|$$

with

$$|\Psi_n\rangle = |\Psi_n^A\rangle|\Psi_n^B\rangle$$

for each n.

$$\hat{\rho}_{sep} = \sum_n p_n \hat{\rho}_A \otimes \hat{\rho}_B$$

- mixture of separable states is always separable
- mixture of entangled states need not be entangled (see example)

Mixture of Bell states

$$\hat{\rho} = \frac{1}{2}|\Phi^+\rangle\langle\Phi^+| + \frac{1}{2}|\Phi^-\rangle\langle\Phi^-| = \frac{1}{2}|00\rangle\langle 00| + \frac{1}{2}|11\rangle\langle 11| =$$

$$\frac{1}{2}\left[|0\rangle\langle 0|_A \otimes |0\rangle\langle 0|_B + |1\rangle\langle 1|_A \otimes |1\rangle\langle 1|_B\right]$$

- the mixed state can be realized by both an ensemble of maximally entangled states and an ensemble of product states
- mixture is a process which destroys entanglement

Example: Werner state

$$\hat{\rho} = \frac{1}{4}(1-\lambda)\hat{Id} + \lambda|\Psi^-\rangle\langle\Psi^-|$$

is entangled for $|\lambda| > 1/3$.

Entanglement concentration, purification and distillation

For various application one needs to have maximally entangled states. How to get such states using only LOCC out of many nonmaximally entangled states?

- Entanglement concentration - how to create maximally entangled pure state from not maximally entangled pure states
- Entanglement purification - how to create maximally entangled pure state from mixed entangled states

Both procedures are sometimes called distillation.

Two general approaches considered:

- global - operate on a large number of copies of a pure or mixed states, i.e. $\hat{\rho}^{\otimes n}$ with $n \longrightarrow \infty$ - theoretically efficient but practically hard to realize
- operates on a small number of copies - not so efficient but experimentally possible

Entanglement measures

- finite regime - for a single copy of a quantum state
- asymptotic regime - for n copies of a quantum state with $n \longrightarrow \infty$

Maximally entangled state in pure bipartite states $\mathcal{H}_d \otimes \mathcal{H}_d$

$$|\Phi_{max}\rangle = \sum_{i=1}^{d} \frac{1}{\sqrt{d}} |\phi_i\rangle \otimes |\phi_i\rangle$$

Entanglement measure $E(\hat{\rho})$ is a real-valued function

$$E: \quad \hat{\rho} \longrightarrow E(\hat{\rho}) \in R,$$

satisfying reasonable postulates:

Entanglement measures postulates

1. **separability**: $E(\hat{\rho}) = 0$ for $\hat{\rho}$ separable
2. **normalization**: $E(\hat{\rho}) = \log_2 d$ for maximally $|\Phi_{max}\rangle$ entangled state
3. **monotonicity**: $E(\hat{\Lambda}\hat{\rho}) \leqslant E(\hat{\rho})$ for any LOCC $\hat{\Lambda}$ [LOCC does not increase entanglement]
4. **continuity**: If $||\hat{\rho} - \hat{\sigma}|| \rightarrow 0$ then $E(\hat{\rho}) - E(\hat{\sigma}) \rightarrow 0$
5. **additivity**: $E(\hat{\rho}^{\otimes n}) = nE(\hat{\rho})$
6. **subadditivity**: $E(\hat{\rho} \otimes \hat{\sigma}) \leqslant E(\hat{\rho}) + E(\hat{\sigma})$
7. **regularization**: $E^{\infty}(\hat{\rho}) = lim_{n\rightarrow\infty} E(\hat{\rho}^{\otimes n})/n$ exists
8. **convexity**: $E(\lambda\hat{\rho} + (1-\lambda)\hat{\sigma}) \leqslant \lambda E(\hat{\rho}) + (1-\lambda)E(\hat{\sigma})$, for $0 \leqslant \lambda \leqslant 1$ [mixing does not increase entanglement]

Pure bipartite states

relative von Neumann entropy $(\hat{\rho} = |\Psi\rangle\langle\Psi|)$

$$E(|\Psi\rangle) = -Tr[\hat{\rho}_A \log_2 \hat{\rho}_A] = -Tr[\hat{\rho}_B \log_2 \hat{\rho}_B]$$

Schmidt rank r $(|\Psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B,\ dim\mathcal{H}_A \leqslant dim\mathcal{H}_B)$

$$|\Psi\rangle = \sum_{i=1}^{r} p_i |\tilde{\Psi}_i^A\rangle|\tilde{\Psi}_i^B\rangle$$

$r \leqslant dim\mathcal{H}_A$ - number of nonzero terms in Schmidt decomposition
(number of entangled degrees of freedom)

Pure bipartite states - example

Single qbit

$$|\phi\rangle = \sum_{ij=1}^{2} \gamma_{ij} |i\rangle_A |j\rangle_B$$

$Tr\gamma\gamma^+ = 1$

$$E(|\phi\rangle) = \mathcal{F}(\frac{1}{2} + \frac{1}{2}\sqrt{1 - 4\det\gamma\gamma^+})$$

where

$\mathcal{F}(x) = -x\log_2 x - (1-x)\log_2(1-x)$

concurrency

$$C = 2\sqrt{\det\gamma\gamma^+}$$

Bell state $\gamma = \sigma_x$
$C = 1$, $E = \mathcal{F}(1/2) = 1$, $r = 2$
Product state $|ii\rangle$
$C = 0$, $E = \mathcal{F}(0) = 0$, $r = 1$

Mixed bipartite states

not completely resolved because of the intricate relation between classical and quantum correlations in mixed states

- E_c - entanglement cost (minimal number of Bell states to create a given state using LOCC) [cont]
- E_D - entanglement of distillation (maximal number of Bell states extracted from a system using LOCC) [cont]
- E_F - entanglement of formation (optimized average von Neumann entropy of reduced density operators for pure states) [add]

$$E_F(\hat{\rho}) = \min_{\{p_i, |\Psi_i\rangle\}} \sum_i p_i S(\hat{\rho}_{i,red})$$

- E_R - relative entropy (distance between entangled $\hat{\rho}$ and the closest separated $\hat{\sigma}$) [add]

$$E_R(\hat{\rho}) = \min_{\hat{\sigma}} [Tr \hat{\rho}(\log_2 \hat{\rho} - \log_2 \hat{\sigma})]$$

  (quite useful)
- many others ...

$$E_D \leqslant E_F \leqslant E_C$$

Multipartite systems

- Quantum mutual information (Total correlation) between the two subsystems $\hat{\rho}_1$ and $\hat{\rho}_2$ of the joint state $\hat{\rho}_{12}$

$$I(\hat{\rho}_1 : \hat{\rho}_2; \hat{\rho}_{12}) = S(\hat{\rho}_1) + S(\hat{\rho}_2) - S(\hat{\rho}_{12})$$

where $S = -Tr\hat{\rho}\log_2 \hat{\rho}$ von Neumann entropy

- Quantum relative entropy between $\hat{\sigma}$ and $\hat{\rho}$

$$S(\hat{\sigma}||\hat{\rho}) = Tr[\hat{\sigma}(\log_2 \hat{\sigma} - \log_2 \hat{\rho})]$$

- Quantum mutual information is a distance of $\hat{\rho}_{12}$ to the closest uncorrelated $\hat{\rho}_1 \otimes \hat{\rho}_2$

$$I(\hat{\rho}_1 : \hat{\rho}_2; \hat{\rho}_{12}) = S(\hat{\rho}_{12}||\hat{\rho}_1 \otimes \hat{\rho}_2)$$

- Multipartite quantum mutual information in $\hat{\rho}$ (generalization)

$$I(\hat{\rho}_1 : \hat{\rho}_2 : \ldots : \hat{\rho}_n; \hat{\rho}) = S(\hat{\rho}||\hat{\rho}_1 \otimes \hat{\rho}_2 \otimes \ldots \otimes \hat{\rho}_n) = \sum_i S(\hat{\rho}_i) - S(\hat{\rho})$$

e.g. $S(\hat{\rho}||\hat{\rho}_{MF}) = \ln Z_{MF} - \ln Z + \beta\langle H_{MF} - H\rangle_H \geqslant 0.$

Entanglement for multipartite system

Relative entanglement

$$E(\hat{\rho}) = min_{\hat{\sigma}\in\{\text{separable}\}} S(\hat{\rho}||\hat{\sigma})$$

the relative entanglement is a distance between $\hat{\rho}$ and the closest classically correlated state

$$E(\hat{\rho}) \leqslant I(\hat{\rho})$$

If we take $\hat{\sigma} = \hat{\rho}_{MF}$ (???)

$$E(\hat{\rho}) = \ln Z_{MF} - \ln Z + \beta\langle H_{MF} - H\rangle_H$$

$$\chi = \frac{\partial^2 \ln Z}{\partial B^2}, \qquad \chi_{sep} - \chi = \frac{\partial^2 E(\hat{\rho})}{\partial B^2} + \beta \frac{\partial^2 \langle H_{MF} - H\rangle_H}{\partial B^2}$$

Th. general bound for multipartite entanglement (Vedral 2003)

$$E(\hat{\rho}) \leqslant \ln Z_{MF} - \ln Z + \beta\langle H_{MF} - H\rangle_H.$$

Entanglement and the III law

Nernst theorem says that $S(T) \to S_0 = const$, or equivalently $C_V(T) = T(\partial S(T)/\partial T)_V \to 0$ when $T \to 0$.

Th. Wiesniak *et al.* (2005): Only if entanglement develops at low temperatures the Nernst theorem is satisfied.

because: separable states give bound for the ground state energy $U(T=0) \geqslant E_B$ and hence for all separable states

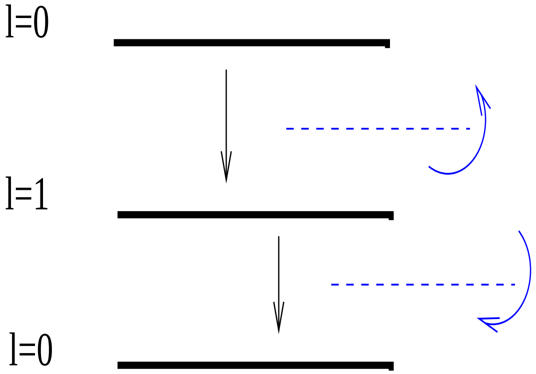
$$C = \frac{\partial U(T)}{\partial T} = \gamma \frac{U(T) - E_0}{T} \geqslant \gamma' \frac{E_B - E_0}{T^{1(2)}}$$

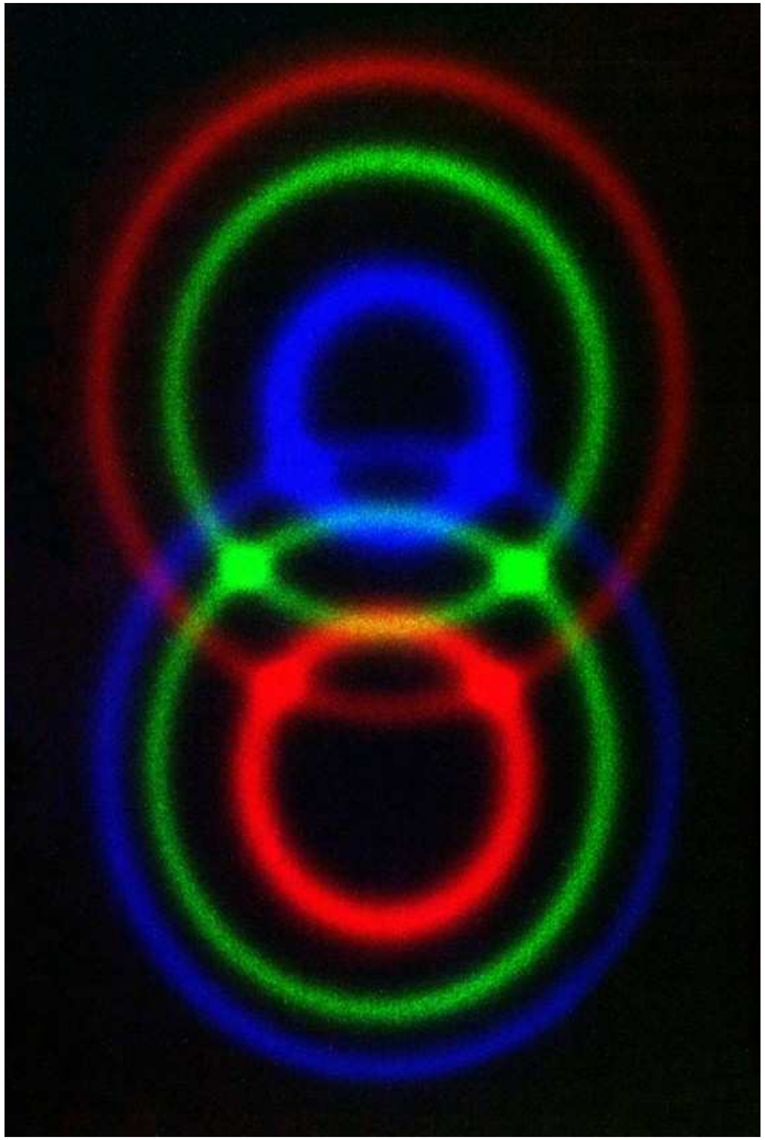
1 for gapless, 2 for gapped systems.
Only when $E_B = E_0$, $C(T) \to 0$
In general $C(T) \to \infty$ for all separable states.

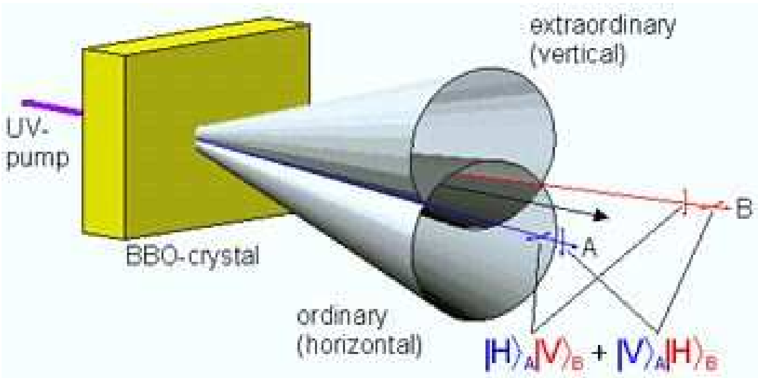
Summary

- Entanglement is a quantum correlation in quantum many body system
- Entanglement does not depend on particular physical representation
- Correlation without correlata
- Entanglement is a resource for certain tasks
- Entanglement can be quantified and measured

Source of entangled photons



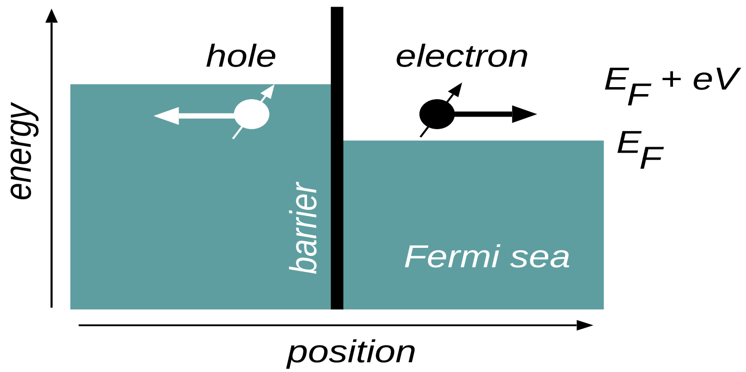


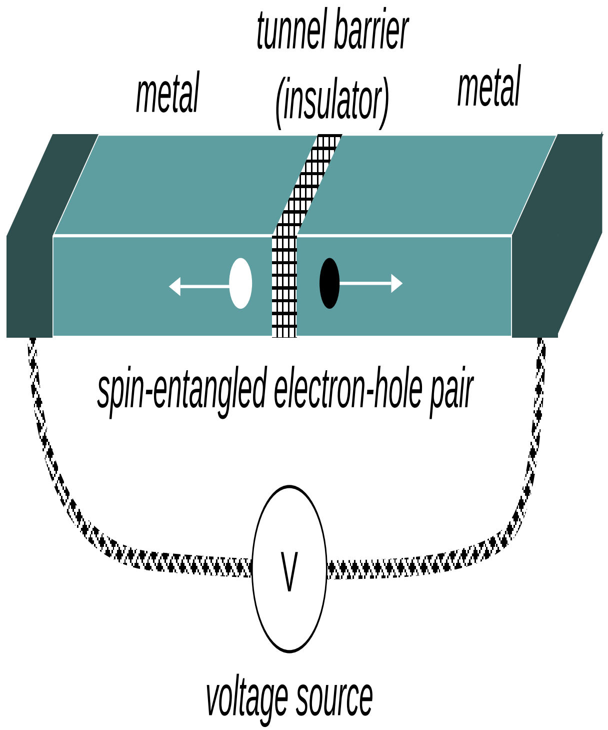


parametric down-conversion, Kwiat *et al.* (1995)

light emitting quantum dots, beam-splitters,...

Source of entangled fermions





Beenakker *et al.* (2003)

$$|\uparrow\rangle_e|\uparrow\rangle_h + e^{i\phi}|\downarrow\rangle_e|\downarrow\rangle_h$$

electron-electron scattering, quantum dots, Cooper pairs, Kondo scattering, ...