Basic definition of superconductors

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Friction, dissipation ...

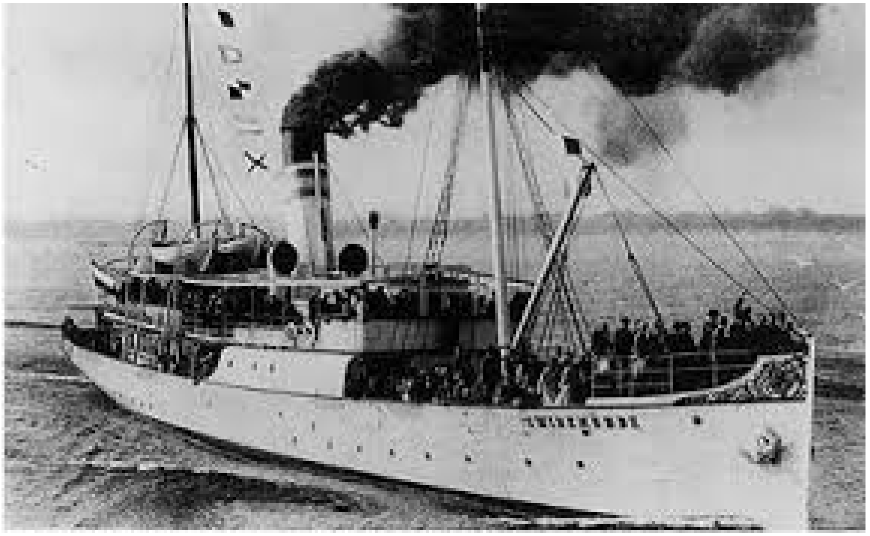
helps...

disturbs...

Friction absorbs energy

electric resistance

$$R \;\; [\Omega = V/A]$$



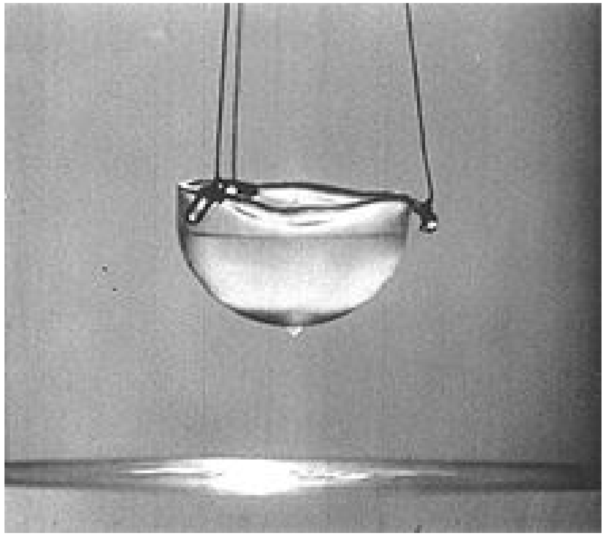
viscosity

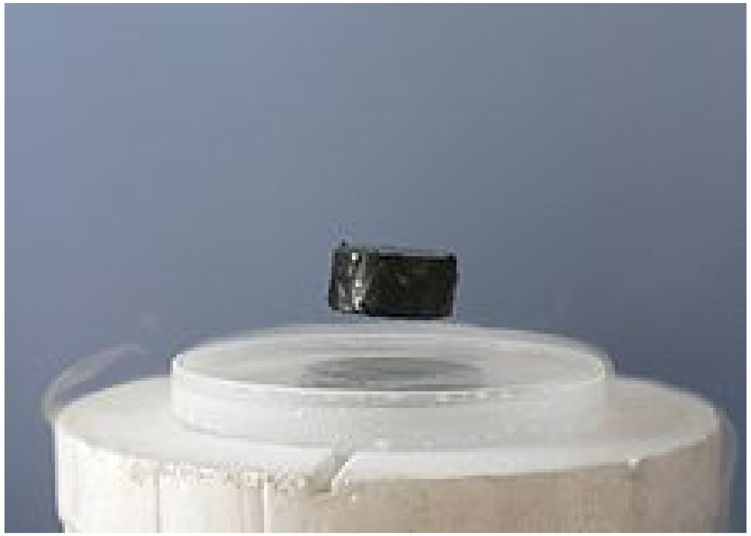
$$\eta \;\; [Pa \cdot s]$$

Two supercases in nature

superfluidity

$$\eta = 0 \; [Pa \cdot s]$$

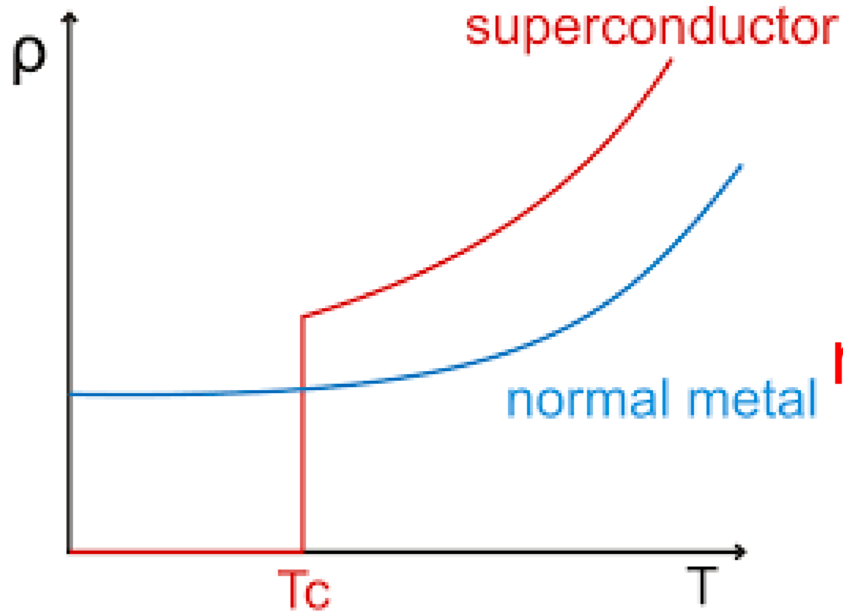




superconductivity

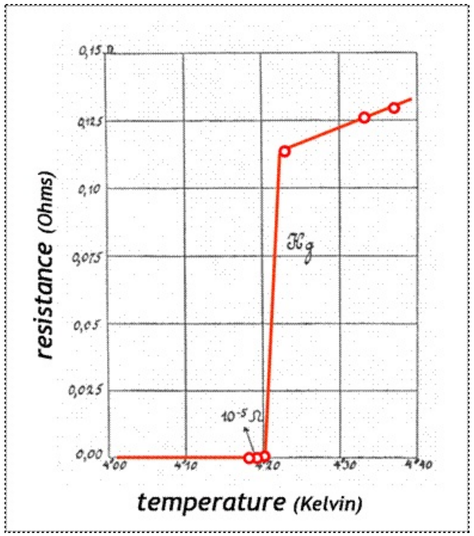
$$R = 0 \; [\Omega]$$

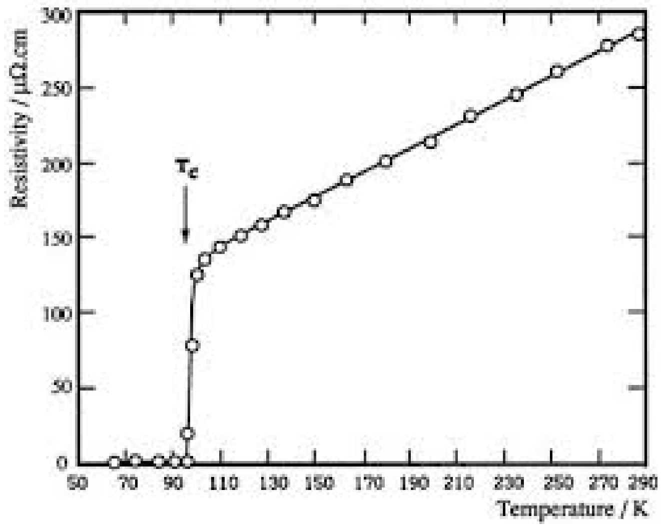
Superconductivity of metals



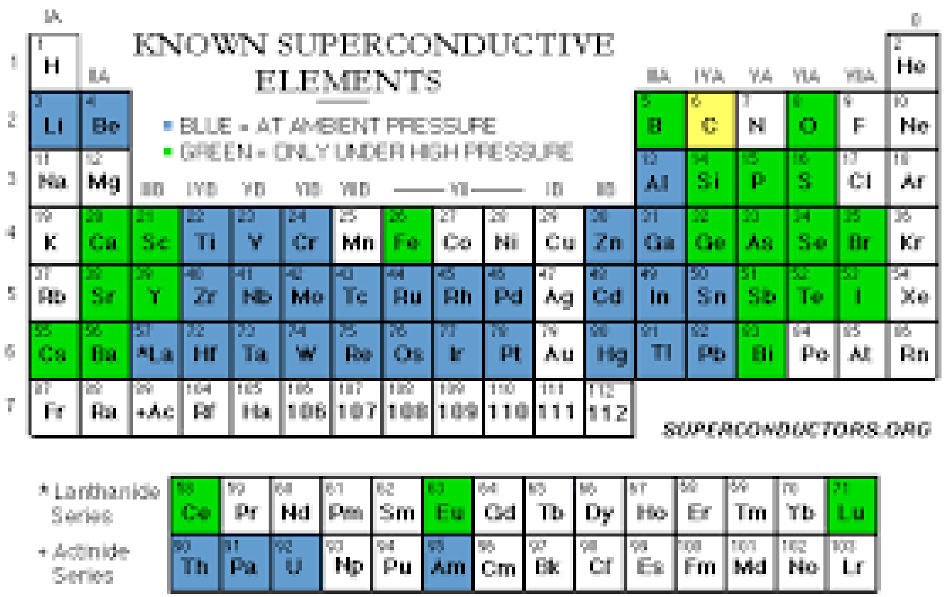
ideal conductor, $R = 0\ [\Omega]$

new stable thermodynamical phase



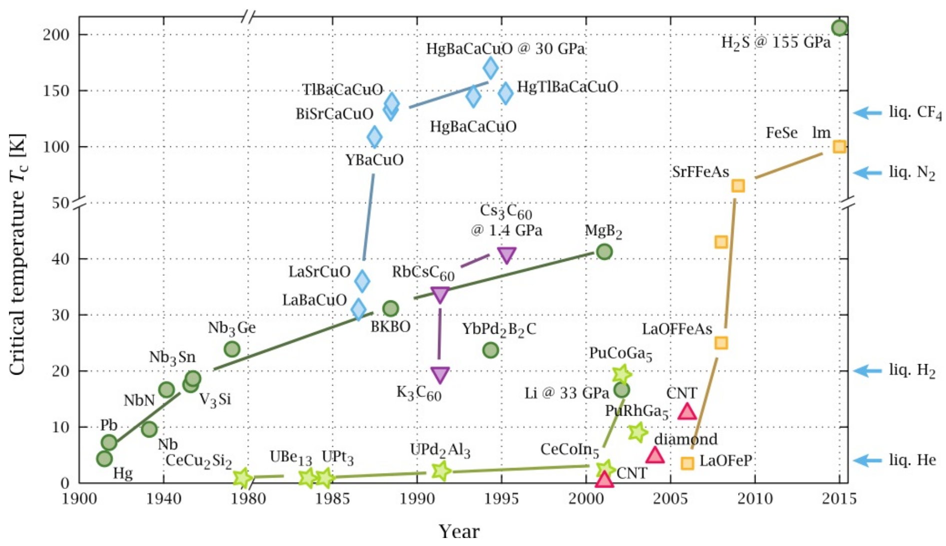


Superconductors



carbonaceous sulfur hydride CH_8S

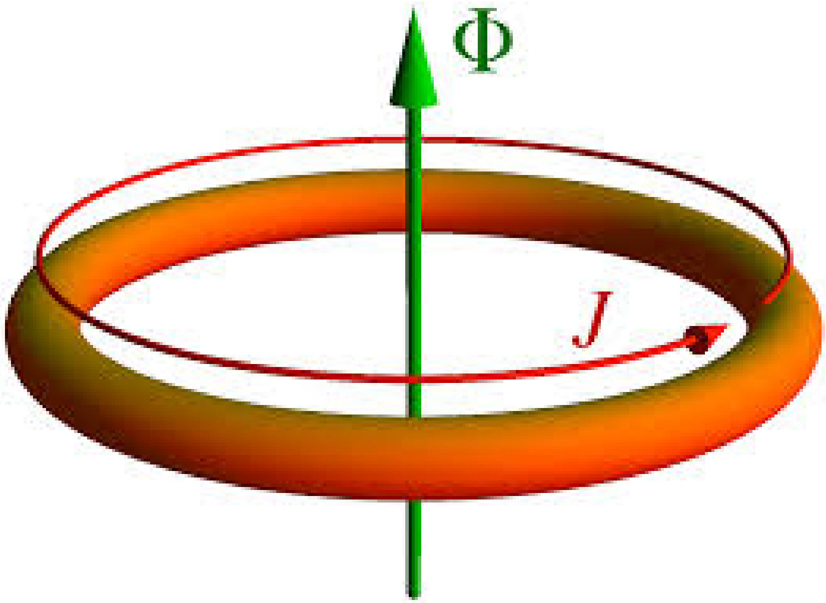
with $T_c \approx +15$C at 267GPa (2020)



www.superconductors.org

https://en.wikipedia.org/wiki/Superconductivity

$R = 0$ - consequences



magnetic flux

$$\Phi = \int \vec{B} \cdot d\vec{s} = BS$$

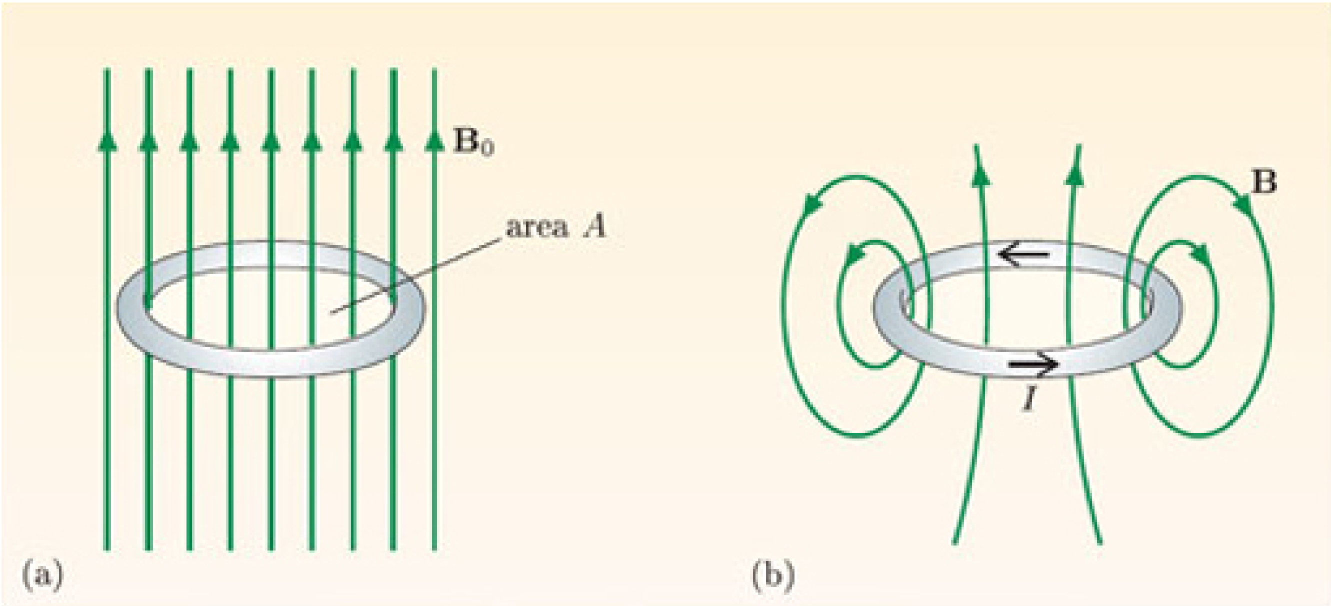
Kirchhoff's law for EMF

$$-S\frac{dB}{dt} = RI + L\frac{dI}{dt}$$

if $R = 0$ then $-S\frac{dB}{dt} = L\frac{dI}{dt}$ hence $\frac{d}{dt}(BS + LI) = 0$

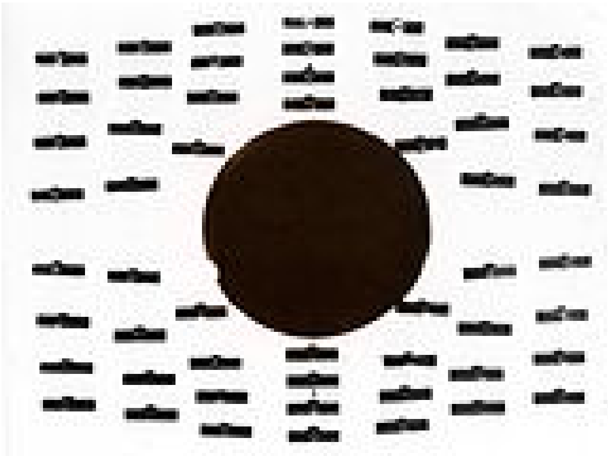
$SB + LI = \mathrm{const}$ - total magnetic flux constant in time

$R = 0$ - consequences



$SB + LI = \mathrm{const}$ - total magnetic flux constant in time

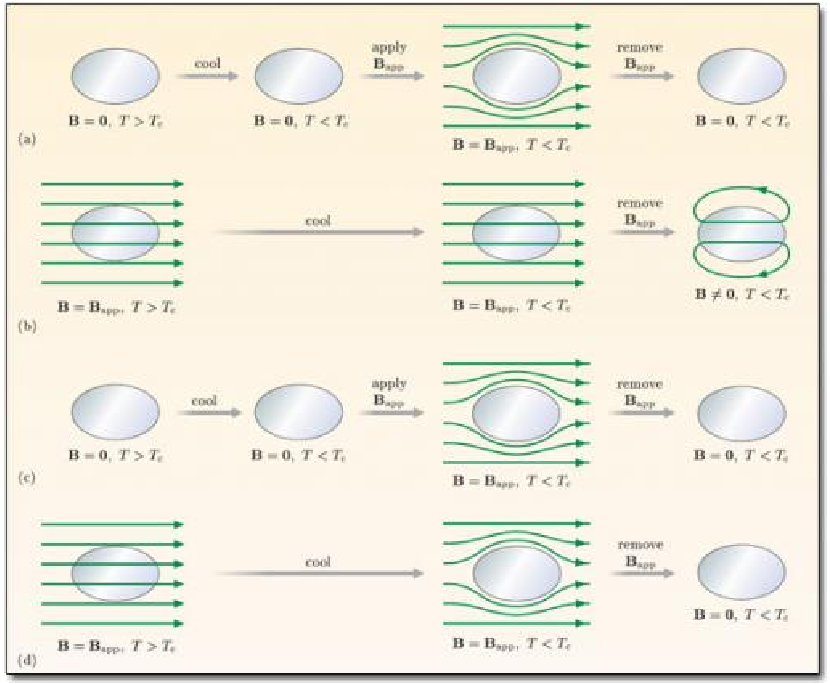
Meissner-Ochsenfeld effect



$\Phi = 0$ **in superconductor!**

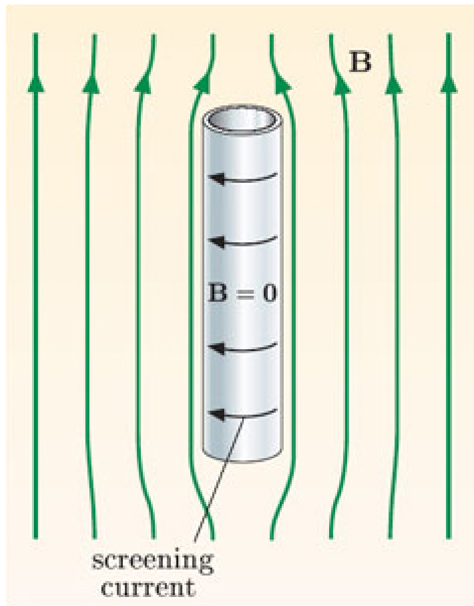
Electrodynamic corrected ... London's equation $\vec{j} = -\frac{n_s e^2}{m}\vec{A}$

Ideal conductor ($R = 0$) vs superconductor



$\Phi = \mathrm{const}$ in ideal conductor

$\Phi = 0$ in superconductor!



magnetic induction vanishes

$$\vec{B} = \mu_0 \vec{H} + \vec{M} = 0$$

supercurrent induces magnetization

Quantization of magnetic flux

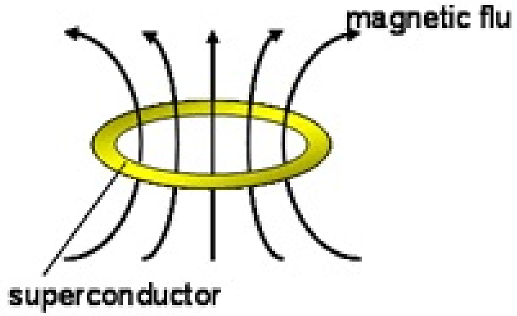


Fig.7 Superconducting ring which hang in the air

$$\Phi = \frac{h}{2e} n, \quad \frac{h}{2e} = 2.07 \cdot 10^{-15}\ [Wb]$$

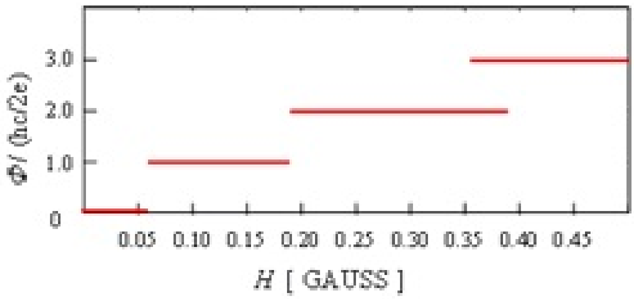


Fig.8 Quantum flux measurement

where $n = 0, 1, 2, ...$

superconductor $=$ ideal conductor $+$ quantized flux

Two types of superconductors

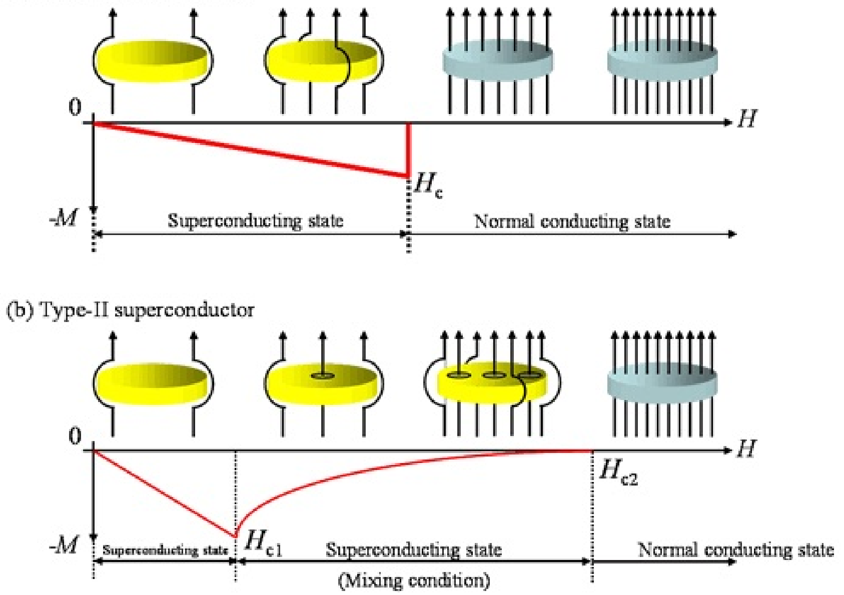
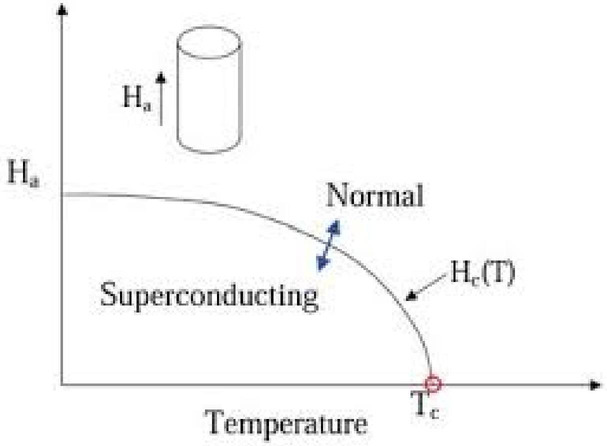
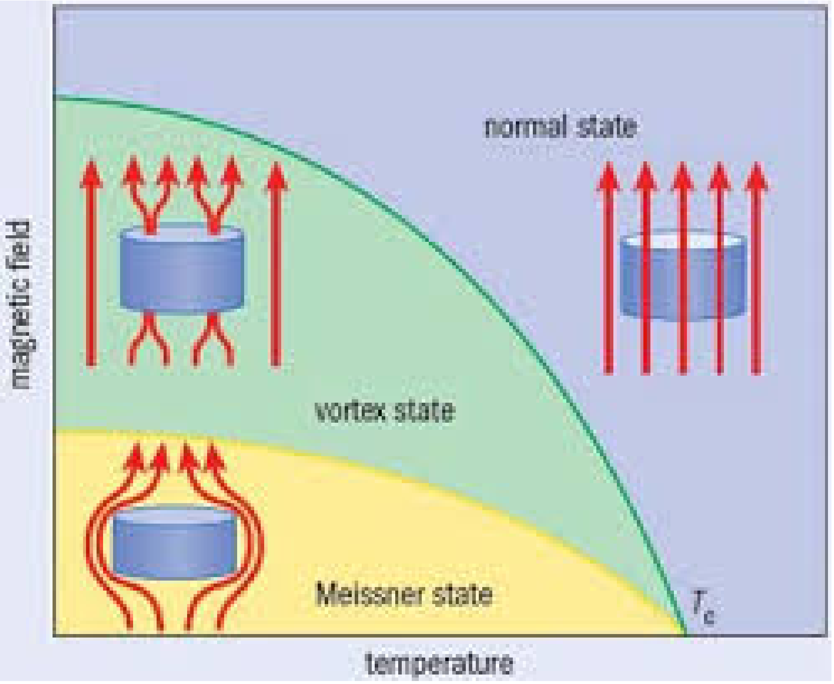


Fig.6 Difference of magnetic field dependence of magnetization between (a) Type-I and (b) Type-II superconductor





Needed a clear definition!

A: Definition based on electromagnetic response

electric field, magnetic induction, vector and scalar potentials

$$\vec{E} = -\frac{\partial \vec{A}}{\partial t} - \vec{\nabla}\phi \quad \text{and} \quad \vec{B} = \vec{\nabla} \times \vec{A}$$

gauge transformation

$$\vec{A} \to \vec{A} + \vec{\nabla}\chi \quad \text{and} \quad \phi \to \phi - \frac{\partial \chi}{\partial t}$$

gauge invariant current (minimal coupling)

$$\vec{j}^{A}(\vec{r}) = \underbrace{\vec{j}(\vec{r})}_{\text{paramagnetic}} \underbrace{- \frac{e}{m}\vec{A}(\vec{r})\rho(\vec{r})}_{\text{diamagnetic}}$$

$$\vec{j}(\vec{r}) = \frac{e}{2m}\sum_{\alpha}(\delta(\vec{r}-\vec{r}_\alpha)\vec{p}_\alpha + \vec{p}_\alpha\delta(\vec{r}-\vec{r}_\alpha)) \quad \text{and} \quad \rho(\vec{r}) = e\sum_{\alpha}\delta(\vec{r}-\vec{r}_\alpha)$$

A: Definition based on electromagnetic response

Perturbation

$$\delta H(t)_A = -\int d_3r \vec{A}(\vec{r},t)\cdot\vec{j}(\vec{r}) \quad \text{and} \quad \delta H(t)_\phi = \int d_3r \phi(\vec{r},t)\rho(\vec{r})$$

Linear response (Kubo) formula

$$\langle j_a^A(\vec{q},\omega)\rangle = [\chi^R_{j_a j_b}(\vec{q},\omega) - \frac{ne^2}{m}\delta_{ab}]A_b(\vec{q},\omega) - \chi^R_{j_a\rho}(\vec{q},\omega)\phi(\vec{q},\omega)$$

where

$$\chi_{O_1O_2}(\vec{r},t;\vec{r}'t') = \frac{i}{\hbar}\theta(t-t')\langle[O_1(\vec{r},t),O_2(\vec{r}',t')]\rangle$$

Not yet a gauge invariant conductivity

$$\langle j_a^A(\vec{q},\omega)\rangle = \sigma_{ab}(\vec{q},\omega)E_b(\vec{q},\omega)$$

A: Definition based on electromagnetic response

Helmholtz theorem

$$\vec{F}(\vec{r}) = \vec{F}^L(\vec{r}) + \vec{F}^T(\vec{r})$$

$\vec{\nabla} \cdot \vec{F}^T(\vec{r}) = 0$ or $\vec{q} \cdot \vec{F}^T(\vec{q}) = 0$ (solenoidal)

$\vec{\nabla} \times \vec{F}^L(\vec{r}) = 0$ or $\vec{q} \times \vec{F}^L(\vec{q}) = 0$ (potential)

transverse conductivity: $E_y(q_x\omega) = i\omega A_y(q_x, \omega)$

$$\langle j_y^A(q_x, \omega) \rangle = \sigma_{yy}^T(q_x, \omega) E_y(q_x, \omega)$$

with

$$\sigma_{yy}^T(q_x, \omega) = \frac{1}{i\omega}[\chi_{j_y j_y}^R(q_x, \omega) - \frac{ne^2}{m}]$$

A: Definition based on electromagnetic response

Longitudinal conductivity:

gauge invariance $\rightarrow$ current conservation

$$\frac{\partial\rho(\vec{q},t)}{\partial t} = -i\vec{q}\cdot\vec{j}(\vec{q},t)$$

hence

$$-i\omega\chi^R_{j_x\rho}(q_x,\omega) = iq_x\frac{ne^2}{m} - iq_x\chi^R_{j_xj_x}(q_x,\omega)$$

and

$$\langle j^A_x(q_x,\omega)\rangle = \frac{1}{i\omega}[\chi^R_{j_xj_x}(q_x,\omega) - \frac{ne^2}{m}](i\omega A_x(q_x,\omega) - iq_x\phi(q_x,\omega)) =$$
$$= [\frac{1}{iq_x}\chi^R_{j_x\rho}(q_x,\omega)](i\omega A_x(q_x,\omega) - iq_x\phi(q_x,\omega))$$

A: Definition based on electromagnetic response

replacing gauge invariant combination of potentials

$$E_x(q_x,\omega) = i\omega A_x(q_x,\omega) - iq_x\phi(q_x,\omega)$$

and from

$$\langle j_x^A(q_x,\omega)\rangle = \sigma_{xx}^L(q_x,\omega)E_x(q_x,\omega)$$

we get longitudinal conductivity

$$\sigma_{xx}^L(q_x,\omega) = \frac{1}{i\omega}[\chi_{j_xj_x}^R(q_x,\omega) - \frac{ne^2}{m}] = [\frac{1}{iq_x}\chi_{j_x\rho}^R(q_x,\omega)]$$

consequence of gauge invariance:

$$\chi_{j_xj_x}^R(q_x,0) - \frac{ne^2}{m} = 0 \quad - \quad \text{thermodynamic response, } \omega = 0 \text{ first}$$

$$\chi_{j_x\rho}^R(0,\omega) = 0$$

A: Definition based on electromagnetic response

uniform DC conductivity ($q_x = 0$ first)

$$\mathrm{Re}\,[\sigma^L_{xx}(q_x = 0, \omega)] = \frac{\mathrm{Im}\chi^R_{j_x j_x}(q_x = 0, \omega)}{\omega} =$$

$$= \mathcal{P}\frac{\mathrm{Im}\chi^R_{j_x j_x}(q_x = 0, \omega)}{\omega} - \pi\delta(\omega)\left[[\mathrm{Re}[\chi^R_{j_x j_x}(q_x = 0, \omega)] - \frac{ne^2}{m}\right]$$

Df. Drude weight

$$D = -\pi \lim_{\omega \to 0}\left[\mathrm{Re}[\chi^R_{j_x j_x}(q_x = 0, \omega)] - \frac{ne^2}{m}\right] \geq 0$$

E.g., free electrons

$$\mathrm{Re}\,[\sigma^L_{xx}(q_x = 0, \omega)] = D\delta(\omega) \quad \text{with} \quad D = \pi\frac{ne^2}{m}$$

A: Definition based on electromagnetic response

W. Kohn's 1960's criteria:

- metal - finite Drude weight at zero temperature
- insulator - zero Drude weight at zero temperature

Metallic behavior is an emergent property in infinite systems, like dissipation (energy level spacing). The limits $q_x \to 0$ and $\omega \to 0$ cannot be exchanged.

In insulators

$$\lim_{\omega \to 0} \lim_{q_x \to 0} \mathrm{Re}\,[\chi^R_{j_x j_x}(q_x, \omega)] = \lim_{q_x \to 0} \lim_{\omega \to 0} \mathrm{Re}\,[\chi^R_{j_x j_x}(q_x, \omega)]$$

Existence of a gap is sufficient but not necessary condition to have an insulator.

A: Definition based on electromagnetic response

What is a superconductor?

London: the wave function of a superconductor is rigid (does not depend on $\vec{A}$); the only possible response is due to the diamagnetic term

$$\langle j_a^T(\vec{q}, \omega = 0) \rangle = -\frac{n_s e^2}{m} A_a^T(\vec{q}, \omega = 0)$$

Hence London's equation

$$\nabla^2 \vec{B} = \frac{n_s e^2}{m} \mu_0 \vec{B},$$

with the solution

$$B_y(x) = B_y(0) e^{-\frac{x}{\lambda}}, \quad \text{where} \quad \lambda = \frac{n_s e^2}{m} \mu_0.$$

A: Definition based on electromagnetic response

What is a superconductor?

Earlier condition (gauge, f-sum rule) on thermodynamic response

$$\chi^R_{j_x j_x}(q_x, 0) - \frac{ne^2}{m} = 0$$

implies no response to a pure vector potential. This is correct only for a longitudinal part of $\vec{A}$. Gauge invariance does not force the transverse response to vanish. Therefore, we can assume

$$\chi^R_{j_y j_y}(q_x, 0) - \frac{ne^2}{m} = -\frac{n_s e^2}{m}$$

with $n_s \leq n$.

Definition: superconductor has a non-vanishing ”transverse Drude weight”!

A: Definition based on electromagnetic response

$$D \equiv \pi \lim_{\omega \to 0} \left[\frac{ne^2}{m} - \mathrm{Re}\,[\chi^R_{j_x j_x}(q_x = 0, \omega)] \right]$$

$$D_S^T \equiv \pi \lim_{q_x \to 0} \left[\frac{ne^2}{m} - \mathrm{Re}\,[\chi^R_{j_y j_y}(q_x, \omega = 0)] \right]$$

$$D_S^L \equiv \pi \lim_{q_x \to 0} \left[\frac{ne^2}{m} - \mathrm{Re}\,[\chi^R_{j_x j_x}(q_x, \omega = 0)] \right]$$

	D	D_S^L	D_S^T
metal	$\neq 0$	0	0
insulator	0	0	0
superconductor	$\neq 0$	0	$\neq 0$

A: Definition based on electromagnetic response

Literature:

R. Schrieffer, *Theory of superconductivity* (1964)

A.-M. Tremblay, *PHY-892 Quantum Material's Theory, from perturbation theory to dynamical-mean field theory* (lecture notes) (2019)

S. Nakajima, Prog. Theor. Phys. **22**, 430 (1959)

The author confesses that he has not made any picture in this presentation, they all have been taken from WWW.

B: Definition based on ODLRO

Off-Diagonal Long Range Order (ODLRO)

single-particle density matrix

$$\rho^{(1)}_{\alpha\alpha'}(\vec{r},\vec{r}') = \langle \psi^\dagger_\alpha(\vec{r})\psi_{\alpha'}(\vec{r}')\rangle$$

two-particle density matrix

$$\rho^{(2)}_{\alpha\beta\alpha'\beta'}(\vec{r}_1,\vec{r}_2,\vec{r}'_1,\vec{r}'_2) = \langle \psi^\dagger_\alpha(\vec{r}_1)\psi^\dagger_\beta(\vec{r}_2)\psi_{\beta'}(\vec{r}'_2)\psi_{\alpha'}(\vec{r}'_1)\rangle$$

or

$$\rho^{(2)}_{\alpha\beta\alpha'\beta'}(\vec{r}_1,\vec{r}_2,\vec{r}'_1,\vec{r}'_2) = N(N-1)\int d_3r_3...d_3r_N \sum_{\gamma_1..\gamma_N}\sum_\nu \frac{e^{-\beta E_\nu}}{Z}$$
$$\times \Psi^*_\nu(\vec{r}_1,\alpha,\vec{r}_2,\beta,\vec{r}_3\gamma_3...)\Psi_\nu(\vec{r}'_1,\alpha',\vec{r}'_2,\beta',\vec{r}_3\gamma_3...)$$

B: Definition based on ODLRO

Spectral decomposition with eigenvalues n_p

$$\rho^{(2)}_{\alpha\beta\alpha'\beta'}(\vec{r}_1, \vec{r}_2, \vec{r}_1', \vec{r}_2') = \sum_p n_p \phi_p^*(\vec{r}_1, \alpha, \vec{r}_2, \beta)\phi_p(\vec{r}_1', \alpha', \vec{r}_2', \beta')$$

ODLRO is such a state where the largest n_0 is of the order N

$$\rho^{(2)}_{\alpha\beta\alpha'\beta'}(\vec{r}_1, \vec{r}_2, \vec{r}_1', \vec{r}_2') \to n_0 \phi_0^*(\vec{r}_1, \alpha, \vec{r}_2, \beta)\phi_0(\vec{r}_1', \alpha', \vec{r}_2', \beta')$$

in the limit $|\vec{r}_i - \vec{r}_i'| \to \infty$ with $|\vec{r}_1 - \vec{r}_2|$ and $|\vec{r}_1' - \vec{r}_2'|$ finite

Theorem: A charge system with ODLRO exhibits a Miessner effect

B: Definition based on ODLRO

Gauge transformation

$$\vec{A}(\vec{r}) \to \vec{A}_0(\vec{r}) + \vec{\nabla}\phi(\vec{r})$$

Take a small space displacement $\vec{r}_j \to \vec{r}_j - \vec{a}$, under which

$$V(r_{jk}) \to V(r_{jk})$$

$$\vec{A}(\vec{r}_j) \to \vec{A}(\vec{r}_j - \vec{a}) = \vec{A}(\vec{r}_j) + \vec{\nabla}_j \underbrace{[\vec{a}\cdot\vec{A}_0(\vec{r}_j) + \phi(\vec{r}_j - \vec{a}) - \phi(\vec{r}_j)]}_{\equiv \chi_{\vec{a}}(\vec{r}_j)}$$

A space displacement induces a gauge transformation in the vector potential.

$$\rho^{(2)}_{\alpha\beta\alpha'\beta'}(\vec{r}_1, \vec{r}_2, \vec{r}\,'_1, \vec{r}\,'_2) = e^{i\frac{e}{\hbar}(\chi_{\vec{a}}(\vec{r}_1)+\chi_{\vec{a}}(\vec{r}_2)-\chi_{\vec{a}}(\vec{r}\,'_1)-\chi_{\vec{a}}(\vec{r}\,'_2))}\rho^{(2)}_{\alpha\beta\alpha'\beta'}(\vec{r}_1-\vec{a}, \vec{r}_2-\vec{a}, \vec{r}\,'_1-\vec{a}, \vec{r}\,'_2-\vec{a})$$

B: Definition based on ODLRO

Factorization hypothesis (ODLRO) implies that

$$\phi_0(\vec{r}_1, \alpha, \vec{r}_2, \beta) = e^{i\xi(\vec{a})} e^{-i\frac{e}{\hbar}(\chi_{\vec{a}}(\vec{r}_1)+\chi_{\vec{a}}(\vec{r}_2))} \phi_0(\vec{r}_1 - \vec{a}, \alpha, \vec{r}_2 - \vec{a}, \beta)$$

Considering two consecutive displacements in alternate order (parallelogram formed by $\vec{a}$ and $\vec{b}$) in a simply connected space yields

$$e^{-i\frac{e}{\hbar}(\chi_{\vec{b}}(\vec{r}_1)+\chi_{\vec{b}}(\vec{r}_2)+\chi_{\vec{a}}(\vec{r}_1-\vec{b})+\chi_{\vec{a}}(\vec{r}_2-\vec{b}))} = e^{-i\frac{e}{\hbar}(\chi_{\vec{a}}(\vec{r}_1)+\chi_{\vec{a}}(\vec{r}_2)+\chi_{\vec{b}}(\vec{r}_1-\vec{a})+\chi_{\vec{b}}(\vec{r}_2-\vec{a}))}$$

since

$$\chi_{\vec{b}}(\vec{r}) + \chi_{\vec{a}}(\vec{r} - \vec{b}) - \chi_{\vec{a}}(\vec{r}) - \chi_{\vec{b}}(\vec{r} - \vec{b}) = \vec{B} \cdot (\vec{a} \times \vec{b})$$

we find

$$\frac{2e}{\hbar} \vec{B} \cdot (\vec{a} \times \vec{b}) = 2\pi n$$

Since $\vec{a}$ and $\vec{b}$ are continuous and n is arbitrary we get $\vec{B} = 0$.

B: Definition based on ODLRO

Zero resistance Hamiltonian depending on macroscopic conjugate variables

$$H = H(n(\vec{r},t), \theta(\vec{r},t))$$

where $n_0 \sim O(N)$ macroscopic eigenstate

$$\phi_0(\vec{r},\alpha) = \sqrt{n(\vec{r},t)}e^{i\theta(\vec{r},t)}$$

Hamiltonian equations $\frac{\partial\theta}{\partial t} = -\frac{\delta H}{\delta n} \quad \text{and} \quad \frac{\partial n}{\partial t} = \frac{\delta H}{\delta \theta}$

Electric potential $U(\vec{r},t) = \delta H / e\delta n(\vec{r},t)$ for stationary currents

$$eU(\vec{r},t) = -\frac{\partial\theta(\vec{r},t)}{\partial t} = 0.$$

From $U = RI$ one gets $R = 0$.

B: Definition based on ODLRO

Literature:

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