

Problem 1.1

Prove the following expansion

$$e^A B e^{-A} = B + [A, B] + \frac{1}{2!} [A, [A, B]] + \frac{1}{3!} [A, [A, [A, B]]] + \dots$$

Prove also the Baker-Hausdorff operator identity

$$e^{A+B} = e^A e^B e^{-\frac{1}{2}[A, B]} = e^{-\frac{1}{2}[A, B]} e^A e^B,$$

holding for operators A and B commuting with $[A, B]$. Finally, prove the general formula,¹

$$e^{t(A+B)} = e^{tA} \text{T exp} \left(\int_0^t d\tau e^{-\tau A} B e^{\tau A} \right),$$

valid for any two operators A and B , in which T denotes the “time” ordered product.

Hints: To prove the expansion solve iteratively the differential equation satisfied by the operator function $C(\lambda) = e^{\lambda A} B e^{-\lambda A}$. Similarly, to prove the Baker-Hausdorff formula consider the function $F(\lambda) = e^{-\lambda B} e^{-\lambda A} e^{\lambda(A+B)}$ and simplify the differential equation it satisfies by using the fact that owing to the assumption, in the expansion of $e^{-\lambda B} A e^{\lambda B}$ in powers of the operator λB only two first terms are nonvanishing and, moreover, $F(\lambda)$ commutes with $A + B$ (show this).

Problem 1.2

Let $|\Psi(t)\rangle_S$ be an eigenvector with the eigenvalue $a(t)$ of the Schrödinger picture operator A^S . Show that $|\Psi\rangle_H$ representing the same state in the Heisenberg picture (defined with respect to $t = 0$) is the eigenvector of $A^H(t)$ with the same eigenvalue $a(t)$. Prove also that if $[A^H(t_0), B^H(t_0)] = C^H(t_0)$, then the same holds for any t .

Problem 1.3

Find the Heisenberg picture operators $\hat{x}_H(t)$ and $\hat{p}_H(t)$ of a particle of mass M moving in one dimension if

- a) it is a free particle ($H = \hat{p}^2/2M$),
- b) $H = \hat{p}^2/2M - \hat{x}F(t)$, where $F(t)$ is an external, time dependent force,
- c) $H = \hat{p}^2/2M + M\omega^2 \hat{x}^2/2$.

In all these cases compute the commutators

$$[\hat{x}_H(t), \hat{x}_H(t')], \quad [\hat{p}_H(t), \hat{p}_H(t')], \quad [\hat{x}_H(t), \hat{p}_H(t')].$$

Using the Heisenberg picture operators compute in cases a) and c) the dispersion of the particle’s position at the instant t expressing it through matrix elements of some combinations of the position and momentum operators at $t = 0$.

¹The Baker-Hausdorff formula is its special case corresponding to $t = 1$ and $[A, [A, B]] = 0$, $[B, [A, B]] = 0$.

Problem 1.4

Justify the identity²

$$a^\dagger a = \sum_{n=0}^{\infty} |n\rangle n \langle n|,$$

in which $|n\rangle$ are the normalized eigenvectors of the operator $a^\dagger a$, where a and a^\dagger are the standard annihilation and creation operators.

Problem 1.5

Find the Heisenberg picture operators $\hat{x}_H(t)$ and $\hat{p}_H(t)$ of the one-dimensional harmonic oscillator the dynamics of which is set by the time dependent Hamiltonian

$$H(t) = \frac{\hat{p}^2}{2M} + \frac{1}{2}M\omega^2\hat{x}^2 - \hat{x}F(t),$$

in which the force $F(t)$ is a given c -number function of time, using the solution³ of the corresponding classical equations of motion with the initial conditions $x(0) = x_0$ and $p(0) = p_0$. To this end, recalling that x_0 and p_0 are also canonical variables related to the standard ones, $x(t)$ and $p(t)$, by the canonical transformation (the generating function of which is just the properly understood action I), promote them to operators \hat{x}_0 and \hat{p}_0 on which the standard commutation rules $[\hat{x}_0, \hat{p}_0] = i\hbar$, etc. are imposed and represent them in the standard way in terms of the creation and annihilation operators. Since the classical Hamiltonian written in terms of the canonical variables x_0 and p_0 vanishes (this is precisely what is ensured by solving the Hamilton-Jacobi equation, but one does not need to do it explicitly here), the operators $\hat{x}(t)$ and $\hat{p}(t)$ obtained from the classical solution in which the operators \hat{x}_0 and \hat{p}_0 are substituted for x_0 and p_0 (expressed, in turn, through the creation and annihilation operators) are just the Heisenberg picture operators. The Heisenberg picture operators $a_H(t)$ and $a_H^\dagger(t)$ can be then read off from the form of $\hat{x}_H(t)$ and $\hat{p}_H(t)$.

A reassuring remark: the description of the problem is long but the steps to do are entirely trivial. After doing it, you will have, perhaps, a better understanding of what “quantization” means.

Problem 1.6

A particle of mass m and electric charge q (in units of $e > 0$) moves in the constant magnetic field $\mathbf{B} = \mathbf{e}_z B$. Find the Heisenberg picture operators $\hat{x}_H(t)$, $\hat{y}_H(t)$ and $\hat{z}_H(t)$ and compute the commutators $[\hat{x}_H(t), \hat{x}_H(t')]$, $[\hat{y}_H(t), \hat{y}_H(t')]$, $[\hat{x}_H(t), \hat{y}_H(t')]$ and $[\hat{x}_H(t), \hat{z}_H(t')]$. Do these commutators depend on the choice of the potential \mathbf{A} (the choice of the gauge)? Consider also the operators $\hat{p}_H^x(t)$, $\hat{p}_H^y(t)$, $\hat{p}_H^z(t)$ and their commutators. Do they depend on the gauge?

Hint: If it is too difficult to work without specifying explicitly a gauge, set e.g. $\mathbf{A} = \mathbf{e}_y \xi B x - \mathbf{e}_x (1 - \xi) B y$ with an arbitrary parameter ξ in order to follow the gauge (in)dependence at least within a restricted class of gauges. To construct the Heisenberg picture

²This is taken from the BMW, but there the problem is formulated with a misprint...

³The solution can be found e.g. in my notes to Classical Mechanics (in polish).

operators $\hat{x}_H(t)$, $\hat{y}_H(t)$, $\hat{p}_H^x(t)$, $\hat{p}_H^y(t)$, take the inspiration from Problem 1.5. Remember that the canonical momenta p^x and p^y are not simply given by $m\dot{x}$ and $m\dot{y}$.

Problem 1.7

A particle of mass M and electric charge q (in units of $e > 0$) moves in the electric and magnetic fields represented by the potentials $\varphi(t, \mathbf{r})$ and $\mathbf{A}(t, \mathbf{r})$. Find the equation of motion satisfied by the Heisenberg picture operator $\hat{\mathbf{r}}_H(t)$, that is compute $d^2\hat{\mathbf{r}}_H(t)/dt^2$. Establish how this derivative differs from the classical formula (written here in the Gauss system of units)

$$M \frac{d^2\mathbf{r}(t)}{dt^2} = qe \left[\mathbf{E}(t, \mathbf{r}) + \frac{\mathbf{v}}{c} \times \mathbf{B}(t, \mathbf{r}) \right].$$

Problem 1.8 (Sucher formula)

Express the difference $E_\Omega - E_{\Omega_0}$ of ground state energies of the Hamiltonians $H = H_0 + \lambda V_{\text{int}}$ and H_0 through the derivative with respect to the parameter λ of the operator⁴

$$S_0^\varepsilon \equiv U_I^{-\varepsilon}(+\infty, 0) U_I^\varepsilon(0, -\infty) = [U_I^{-\varepsilon}(0, +\infty,)]^\dagger U_I^\varepsilon(0, -\infty),$$

that is, prove the so-called Sucher formula

$$E_\Omega - E_{\Omega_0} = \frac{1}{2} i\hbar \varepsilon \lambda \frac{\partial}{\partial \lambda} \ln \langle \Omega_0 | S_0^\varepsilon | \Omega_0 \rangle,$$

Problem 1.9

By considering the differential equation satisfied by it, find the complete evolution operator $U^\varepsilon(t, 0)$, including its phase, corresponding to the Gell-Mann - Low modification $V_{\text{int}} \rightarrow e^{\varepsilon t} V_{\text{int}}$ of the Hamiltonian ($\Delta_\omega = \hbar\omega/2$)

$$H = H_0 + V_{\text{int}} = \hbar\omega a^\dagger a + \Delta_\omega + \lambda a^\dagger + \lambda^* a,$$

of the linearly perturbed harmonic oscillator. Show then by an explicit computation that the expression

$$\lim_{T \rightarrow \infty} \left(\lim_{\varepsilon \rightarrow 0^+} \langle \Omega_0 | U_I^{-\varepsilon}(T, 0) [U_I^\varepsilon(-T, 0)]^\dagger | \Omega_0 \rangle \right) \equiv \lim_{T \rightarrow \infty} \langle \Omega_0 | U_I(T, -T) | \Omega_0 \rangle,$$

in which $U_I^{\pm\varepsilon}(t, 0)$ are the interaction picture evolution operators corresponding to the interaction term adiabatically switched on and off and the limit $\varepsilon \rightarrow 0^+$ is taken first, behaves as

$$\exp \left\{ -i \frac{2T}{\hbar} (E_\Omega - E_{\Omega_0}) \right\}.$$

⁴ $U_I^{-\varepsilon}(+\infty, 0)$ is the interaction picture evolution operator corresponding to replacing the original time independent interaction λV_{int} by $\lambda V_{\text{int}} e^{-\varepsilon t}$. The limit $\varepsilon \rightarrow 0^+$ is implicit.

Hint: In order to ensure the proper transformation to the Heisenberg picture of the basic operators a and a^\dagger , the sought evolution operator must have the form

$$U^\varepsilon(t, 0) = e^{i\varphi(t)} e^{-iH_0 t/\hbar} e^{h(t)} a^\dagger e^{-h^*(t)} a, \quad h(t) = -\frac{i}{\hbar} \int_0^t d\tau \lambda e^{(\varepsilon+i\omega)\tau},$$

so only the phase $\varphi(t)$ has to be determined.

Problem 1.10 (Thomas - Reiche - Kuhn sum rule)

The Hamiltonian of a set of N *identical and indistinguishable* nonrelativistic spinless particles of mass M has the general form

$$H = \sum_{a=1}^N \frac{\mathbf{p}_a^2}{2M} + V(\mathbf{r}_1, \dots, \mathbf{r}_N).$$

Defining the operators

$$O_r(\mathbf{a}) = O_r^\dagger(\mathbf{a}) = \sum_{a=1}^N \mathbf{a} \cdot \hat{\mathbf{r}}_a, \quad O_p(\mathbf{a}) = O_p^\dagger(\mathbf{a}) = \sum_{a=1}^N \mathbf{a} \cdot \hat{\mathbf{p}}_a,$$

in which \mathbf{a} can be any (real for Hermiticity) vector, prove that if $|s\rangle$ is a normalizable (and normalized to unity) eigenvector of H , the following sum rules

$$\begin{aligned} \sum_l |\langle l | O_p(\mathbf{a}) | s \rangle|^2 &= \frac{M^2}{\hbar^2} \sum_l (E_l - E_s)^2 |\langle l | O_r(\mathbf{a}) | s \rangle|^2, \\ \sum_l (E_l - E_s) |\langle l | O_r(\mathbf{n}) | s \rangle|^2 &= \frac{N\hbar^2}{2M}, \\ \sum_l (E_l - E_s) |\langle l | e^{iO_r(\mathbf{q})} | s \rangle|^2 &= N \frac{\hbar^2 \mathbf{q}^2}{2M}, \end{aligned}$$

hold. The second one, in which it is assumed that $\mathbf{n}^2 = 1$, is called the Thomas-Reiche-Kuhn sum rule. The summations over l , where $|l\rangle$ are eigenvectors of H , mean also integrations over the continuous part of the Hamiltonian spectrum.

Hint: Prove first the identity $O_p(\mathbf{a}) = i(M/\hbar)[H, O_r(\mathbf{a})]$. To prove the TRK rule compute in two ways the $|s\rangle$ state expectation value of the double commutator $[[H, O_r(\mathbf{n})], O_r(\mathbf{n})]$ and to prove the last one work out the operator $e^{-iO_r(\mathbf{q})} H e^{iO_r(\mathbf{q})} - H$ using the expansion proved in Problem 1.1 and take the expectation value of both sides in the normalized eigenvector $|s\rangle$ of H .