Problem 3.1
Prove that the Wightman function $\Delta_+(x, m)$ defined by the integral

$$\Delta_+(x,m) = \int d\Gamma_{\mathbf{k}}\, e^{-ik\cdot x} \equiv \int \frac{d^3\mathbf{k}}{(2\pi)^3 2E(\mathbf{k},m)}\, e^{-ik_\mu x^\mu},$$

in which $E(\mathbf{k}, m) = \sqrt{\mathbf{k}^2 + m^2}$ is, for $x^2 < 0$, an even function of x^μ. Explain also why the same argument does not ensure vanishing for $(x-y)^2 < 0$ of the commutator of the (interaction picture) field operators $A^i(x)$ and $A^j(y)$ constructed out of the creation and annihilation operators of a massless spin 1 particle (having two helicity states); only its vanishing for $x^0 = y^0$ is secured.
Hint: Use the Lorentz invariance of the integral defining $\Delta_+(x) \equiv \Delta_+(x, m)$.

Problem 3.2
The functions $u_l(\mathbf{p}, \sigma)$ and $v_l(\mathbf{p}, \sigma)$ corresponding to definite spin projections on the z-axis in the rest frame of a massive spin s particle, which enter the free field (interaction picture) operators

$$\phi_l^{(+)}(x) = \int d\Gamma_{\mathbf{p}} \sum_\sigma u_l(\mathbf{p},\sigma)\, e^{-ip\cdot x}\, a(\mathbf{p},\sigma)\,,$$
$$\phi_l^{(-)}(x) = \int d\Gamma_{\mathbf{p}} \sum_\sigma v_l(\mathbf{p},\sigma)\, e^{+ip\cdot x}\, a^\dagger(\mathbf{p},\sigma)\,,$$

(the indices l transform according to some regular representation $D_{lk}(\Lambda)$ of the Lorentz group) and appear in Feynman rules for initial and and final state of the particle, are given in the general case by the formulae

$$u_l(\mathbf{p},\sigma) = \sum_k D_{lk}(L_p)\, u_k(\mathbf{0},\sigma)\,,$$
$$v_l(\mathbf{p},\sigma) = \sum_k D_{lk}(L_p)\, v_k(\mathbf{0},\sigma)\,,$$

in which L_p is the standard Lorentz transformation (see Problem 2.18). Using the results of Problem 2.25 give the analogous general formulae for the functions $u_l(\mathbf{p}, \lambda)$ and $v_l(\mathbf{p}, \lambda)$ corresponding to a massive particle of definite helicity (spin projection onto the particle's momentum).

Problem 3.3
Construct explicitly the "wave functions" $u_l(\mathbf{p}, \sigma)$ and $v_l(\mathbf{p}, \sigma)$ entering the (free) field operator $V^\mu(x)$ transforming as a Lorentz vector and constructed out of the creation and annihilation operators of a massive spin 1 particle with definite spin projection σ onto the z-axis (in the particle's rest frame). Construct also explicitly the functions $u_l(\mathbf{p}, \lambda)$ and $v_l(\mathbf{p}, \lambda)$ corresponding to a definite helicity λ of such a particle. (All these u_l and v_l functions corresponding to massive spin 1 particles are usually denoted ϵ^μ and $\epsilon^{\mu*}$). Show

that the field operator $V^\mu(x)$ written in terms of the creation and annihilation operators corresponding to definite helicities also satisfies the relations

$$\mathcal{P}V^\mu(x)\mathcal{P}^{-1} = -\eta^* P^\mu_{\ \nu} V^\nu(P\cdot x) \quad \text{and} \quad \mathcal{T}V^\mu(x)\mathcal{T}^{-1} = -\zeta^* T^\mu_{\ \nu} V^\nu(T\cdot x)\,.$$

Problem 3.4
Using the explicit zero momentum forms of the u and v functions (spinors) corresponding to a mass m spin 1/2 particle (fermion) with the spin projection σ onto the z-axis show that

$$\begin{aligned}\bar{u}(\mathbf{p},\sigma)\cdot u(\mathbf{p},\sigma') &= \ \ 2m\,\delta_{\sigma\sigma'}\,,\\ \bar{v}(\mathbf{p},\sigma)\cdot v(\mathbf{p},\sigma') &= -2m\,\delta_{\sigma\sigma'}\,,\\ \bar{u}(\mathbf{p},\sigma)\cdot v(\mathbf{p},\sigma') = \bar{v}(\mathbf{p},\sigma')\cdot u(\mathbf{p},\sigma') &= 0\,,\end{aligned}$$

and that ($E_\mathbf{p} = \sqrt{\mathbf{p}^2+m^2}$)

$$\begin{aligned}u^\dagger(\mathbf{p},\sigma)\cdot u(\mathbf{p},\sigma') &= 2E_\mathbf{p}\,\delta_{\sigma\sigma'}\,,\\ v^\dagger(\mathbf{p},\sigma)\cdot v(\mathbf{p},\sigma') &= 2E_\mathbf{p}\,\delta_{\sigma\sigma'}\,,\\ u^\dagger(\mathbf{p},\sigma)\cdot v(-\mathbf{p},\sigma') = v^\dagger(-\mathbf{p},\sigma)\cdot u(\mathbf{p},\sigma') &= 0\,.\end{aligned}$$

Consider both, Dirac and chiral (Weyl) representations. Prove similar relations satisfied by the helicity spinors $u(\mathbf{p},\lambda)$ and $v(\mathbf{p},\lambda)$ constructed in Problem 3.2.

Problem 3.5
Construct explicitly the spinors $u(\mathbf{p},\sigma=\pm\frac{1}{2})$ and $v(\mathbf{p},\sigma=\pm\frac{1}{2})$ corresponding to mass m spin 1/2 fermions using the appropriate Lorentz transformation. Give the formulae in both, Dirac and Weyl (chiral), representations of the gamma matrices. Check that up to a phase factor ($\not{p} \equiv p_\mu\gamma^\mu$)

$$\begin{aligned}u(\mathbf{p},\sigma) &= \frac{\not{p}+m}{\sqrt{2m(E_\mathbf{p}+m)}}\,u(\mathbf{0},\sigma)\,,\\ v(\mathbf{p},\sigma) &= \frac{\not{p}-m}{\sqrt{2m(E_\mathbf{p}+m)}}\,v(\mathbf{0},\sigma)\,.\end{aligned}$$

Problem 3.6
Construct explicitly the spinors $u(\mathbf{p},\lambda=\pm\frac{1}{2})$, and $v(\mathbf{p},\lambda=\pm\frac{1}{2})$, corresponding to mass m spin 1/2 fermions of definite helicity, using the appropriate Lorentz transformation (Problem 3.2). Give the formulae in both, Dirac and Weyl (chiral), representations of the gamma matrices.

Problem 3.7
Consider the field operator $\psi(x) = \kappa_+\psi^{(+)}(x) + \kappa_-\psi^{(-)}(x)$ associated with a spin 1/2 fermion which is not ascribed any conserved charges (it is the so-called Majorana fermion

which is its own antiparticle) with

$$\psi^{(+)}(x) = \int d\Gamma_{\mathbf{p}} \sum_{\sigma} u(\mathbf{p},\sigma)\, e^{-ip\cdot x}\, b(\mathbf{p},\sigma)\,,$$
$$\psi^{(-)}(x) = \int d\Gamma_{\mathbf{p}} \sum_{\sigma} v(\mathbf{p},\sigma)\, e^{+ip\cdot x}\, b^{\dagger}(\mathbf{p},\sigma)\,.$$

Show that the local causality requires such particles to be fermions i.e. that it is the anticommutators: $\{\psi(x),\ \psi^{\dagger}(y)\}$, $\{\psi(x),\ \psi(y)\}$ and $\{\psi^{\dagger}(x),\ \psi^{\dagger}(y)\}$ which can be made to vanish for $(x-y)^2 < 0$, provided $|\kappa_+| = |\kappa_-|$ and $b_u = -b_v$ (recall that $b_u = \pm 1$ and $b_v = \pm 1$ are the eigenvalues of the $\beta = \gamma^0$ matrix on the spinors $u(\mathbf{0},\sigma)$ and $v(\mathbf{0},\sigma)$, respectively). Show that the intrinsic parity of such a Majorana fermion must be $\pm i$.

Problem 3.8
Show that the operators

$$\psi_{\alpha}(x)\,(C^{-1})_{\alpha\beta}\,\psi_{\beta}(x)\,, \qquad \text{and} \qquad \bar{\psi}_{\alpha}(x)\,C_{\alpha\beta}\,\bar{\psi}_{\beta}(x)\,,$$

constructed out of the Majorana spinor field operators constructed in Problem 3.7 associated with a neutral massive spin $\frac{1}{2}$ particle are Lorentz scalars.

Problem 3.9
Let us introduce the Feynman's notation: $\not a \equiv \gamma^{\mu} a_{\mu}$, where a_{μ} is an arbitrary four-vector. Using the trace property tr(AB)=tr(BA) and the defining relation $\{\gamma^{\mu},\gamma^{\nu}\} = 2g^{\mu\nu}$ show that the following relations hold:

$$\begin{aligned}
&\mathrm{tr}(\not a) = 0\,,\\
&\mathrm{tr}(\not a\ \not b) = 4a_{\mu}b^{\mu}\,,\\
&\mathrm{tr}(\not a_1\ \not a_2 \cdots \not a_{2k+1}) = 0\,,\\
&\mathrm{tr}(\not a\ \not b\ \not c\ \not d) = 4\left[(a\cdot b)(c\cdot d) + (a\cdot d)(c\cdot b) - (a\cdot c)(b\cdot d)\right],\\
&\mathrm{tr}(\gamma^5) = 0\,,\\
&\mathrm{tr}(\not a\ \gamma^5) = 0\,,\\
&\mathrm{tr}(\not a\ \not b\,\gamma^5) = 0\,,\\
&\mathrm{tr}(\not a\ \not b\ \not c\,\gamma^5) = 0\,,\\
&\mathrm{tr}(\not a\ \not b\ \not c\ \not d\ \gamma^5) = -4i\,\epsilon^{\mu\nu\lambda\rho} a_{\mu}b_{\nu}c_{\lambda}d_{\rho}\,.
\end{aligned}$$

The conventions are: $\epsilon_{0123} = -1$ and $\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = -(i/4!)\,\epsilon_{\mu\nu\lambda\rho}\,\gamma^{\mu}\gamma^{\nu}\gamma^{\lambda}\gamma^{\rho}$.

Problem 3.10
Show that

$$\begin{aligned}
&\bar{u}(\mathbf{p}_2,\sigma_2)\left[(p_2+p_1)^{\mu}\mathbf{P} + i\sigma^{\mu\nu}(p_2-p_1)_{\nu}\mathbf{P}\right] u(\mathbf{p}_1,\sigma_1)\\
&\qquad\qquad = \bar{u}(\mathbf{p}_2,\sigma_2)\left(m_2\gamma^{\mu}\mathbf{P} + m_1\mathbf{P}\gamma^{\mu}\right) u(\mathbf{p}_1,\sigma_1)\,,
\end{aligned}$$

where $p_1^2 = p_2^2 = m_2^2$, $\sigma^{\mu\nu} = (i/2)[\gamma^\mu,\ \gamma^\nu]$ and $\mathbf{P}$ can be I, γ^5 or one of the projectors $\mathbf{P}_L$, $\mathbf{P}_R$ defined as

$$\mathbf{P}_L = \frac{1}{2}(I - \gamma^5)\,, \qquad\qquad \mathbf{P}_R = \frac{1}{2}(I + \gamma^5)\,.$$

Use these results to obtain the standard Gordon's identity

$$\bar{u}(\mathbf{p}_2, \sigma_2)\gamma^\mu u(\mathbf{p}_1, \sigma_1) = \bar{u}(\mathbf{p}_2, \sigma_2)\left[\frac{(p_2 + p_1)^\mu}{2m} + \frac{i}{2m}\,\sigma^{\mu\nu}(p_2 - p_1)_\nu\right] u(\mathbf{p}_1, \sigma_1)\,,$$

in which $p_1^2 = m^2$, $p_2^2 = m^2$.
Hint: Use the fact that

$$\bar{u}_2\mathbf{A}(\not{p}_1 - m_1)\, u_1 + \bar{u}_2\,(\not{p}_2 - m_2)\mathbf{B}\, u_1 = 0 + 0 = 0\,,$$

for any two matrices $\mathbf{A}$ and $\mathbf{B}$ in the spinor space.

Problem 3.11
Express $\epsilon^{\mu\nu\lambda\rho}\gamma_\rho$ through the gamma matrices (without the epsilon tensor). Find also a corresponding representation of $\epsilon^{\mu\nu\lambda\rho}\gamma_\lambda\gamma_\rho$. Use the second result to prove that

$$\sigma^{\mu\nu}\gamma^5 \otimes \sigma_{\mu\nu}\gamma^5 = \sigma^{\mu\nu} \otimes \sigma_{\mu\nu}\,,$$
$$\sigma^{\mu\nu}\gamma^5 \otimes \sigma_{\mu\nu} = \sigma^{\mu\nu} \otimes \sigma_{\mu\nu}\gamma^5\,,$$

i.e. that $(\sigma^{\mu\nu}\gamma^5)_{\alpha\beta}(\sigma_{\mu\nu}\gamma^5)_{\alpha'\beta'} = (\sigma^{\mu\nu})_{\alpha\beta}(\sigma_{\mu\nu})_{\alpha'\beta'}$, etc.

Problem 3.12
Show that if the algebra of the gamma matrices formally corresponds to d-dimensional space-time, hold the following relations

$$\gamma^\mu\gamma_\mu = d\,,$$
$$\gamma^\mu\ \not{a}\ \gamma_\mu = (2-d)\ \not{a}\,,$$
$$\gamma^\mu\ \not{a}\ \not{b}\ \gamma_\mu = 4\ a_\mu b^\mu + (d-4)\ \not{a}\ \not{b},$$
$$\gamma^\mu\ \not{a}\ \not{b}\ \not{c}\ \gamma_\mu = -2\ \not{c}\ \not{b}\ \not{a} + (4-d)\ \not{a}\ \not{b}\ \not{c},$$
$$\gamma^\mu\ \not{a}\ \not{b}\ \not{c}\ \not{d}\ \gamma_\mu = 2\ \not{d}\ \not{a}\ \not{b}\ \not{c} + 2\ \not{c}\ \not{b}\ \not{a}\ \not{d} + (d-4)\ \not{a}\ \not{b}\ \not{c}\ \not{d}\,.$$

Problem 3.13
Derive the general formula allowing to decompose products of four spinors u_1, $\bar{u}_2$, u_3, $\bar{u}_4$ of the generic form

$$[\bar{u}_4\Gamma u_3]\,[\bar{u}_2\Gamma' u_1]\,,$$

in which Γ and Γ' are two arbitrary matrices, into the sum of products

$$\left[\bar{u}_4\Gamma^M u_1\right]\,\left[\bar{u}_2\Gamma^N u_3\right]\,,$$

where Γ^M form a complete basis of matrices in the spinor space. Write down explicitly rearrangements of the five basic structures

$$[\bar{u}_4 u_3]\,[\bar{u}_2 u_1]\,, \qquad [\bar{u}_4\gamma^\mu u_3]\,[\bar{u}_2\gamma_\mu u_1]\,, \qquad [\bar{u}_4\sigma^{\mu\nu} u_3]\,[\bar{u}_2\sigma_{\mu\nu} u_1]\,,$$
$$\left[\bar{u}_4\gamma^\mu\gamma^5 u_3\right]\left[\bar{u}_2\gamma_\mu\gamma^5 u_1\right], \qquad \left[\bar{u}_4\gamma^5 u_3\right]\left[\bar{u}_2\gamma^5 u_1\right].$$

Hint: Take for the basis Γ^M the following sixteen linearly independent matrices

$$1\,, \qquad \gamma^\mu\,, \qquad \sigma^{\lambda\rho}\,, \qquad \gamma^\nu\gamma^5\,, \qquad i\gamma^5\,.$$

Problem 3.14
Using the general formulae derived in Problem 3.13 write down the Fierz rearrangements of the following structures:

$$[\bar{u}_4\gamma^\mu \mathbf{P}_L u_3][\bar{u}_2\gamma_\mu \mathbf{P}_L u_1]\,, \qquad [\bar{u}_4\gamma^\mu \mathbf{P}_R u_3][\bar{u}_2\gamma_\mu \mathbf{P}_R u_1]\,, \qquad [\bar{u}_4\gamma^\mu \mathbf{P}_L u_3][\bar{u}_2\gamma_\mu \mathbf{P}_R u_1]\,,$$

and

$$[\bar{u}_4\mathbf{P}_L u_3][\bar{u}_2\mathbf{P}_L u_1]\,, \qquad [\bar{u}_4\mathbf{P}_R u_3][\bar{u}_2\mathbf{P}_R u_1]\,, \qquad [\bar{u}_4\mathbf{P}_R u_3][\bar{u}_2\mathbf{P}_L u_1]\,.$$

Write down also the corresponding identities for the corresponding spinor matrices (i.e. the same identities but written without spinors). Using these result express the product $[\bar{u}_4\sigma^{\mu\nu}u_3][\bar{u}_2\sigma_{\mu\nu}\mathbf{P}_L u_1]$ through $[\bar{u}_4\sigma^{\mu\nu}u_1][\bar{u}_2\sigma_{\mu\nu}\mathbf{P}_L u_3]$ and $[\bar{u}_4\mathbf{P}_L u_1][\bar{u}_2\mathbf{P}_L u_3]$ (and the product $[\bar{u}_4\sigma^{\mu\nu}u_3][\bar{u}_2\sigma_{\mu\nu}\mathbf{P}_R u_1]$).

Problem 3.15
Check that if λ_α transforms as $\lambda'_\alpha = M_\alpha{}^\beta\lambda_\beta$ under Lorentz (or, more precisely, SL(2,C)) transformations, then $\lambda^\alpha = \epsilon^{\alpha\beta}\lambda_\beta$ transforms with the matrix M^{T-1}, i.e. $\lambda'^\alpha = (M^{T-1})^\alpha{}_\beta\lambda^\beta$. Similarly, show that if $\bar{\chi}'_{\dot{\alpha}} = (M^*)_{\dot{\alpha}}{}^{\dot{\beta}}\bar{\chi}_{\dot{\beta}}$ then $\bar{\chi}^{\dot{\alpha}} = \bar{\chi}_{\dot{\beta}}\,\bar{\epsilon}^{\dot{\beta}\dot{\alpha}}$ transforms as $\bar{\chi}'^{\dot{\alpha}} = (M^{\dagger-1})^{\dot{\alpha}}{}_{\dot{\beta}}\,\bar{\chi}^{\dot{\beta}}$. Use the convention

$$\epsilon^{12} = -\epsilon^{21} = -1\,, \qquad \bar{\epsilon}^{\dot{1}\dot{2}} = -\bar{\epsilon}^{\dot{2}\dot{1}} = 1\,,$$
$$\epsilon_{12} = -\epsilon_{21} = -1\,, \qquad \bar{\epsilon}_{\dot{1}\dot{2}} = -\bar{\epsilon}_{\dot{2}\dot{1}} = 1\,.$$

Problem 3.16
Prove by direct calculation the following identities

$$(\sigma^\mu)_{\alpha\dot{\beta}}\,(\bar{\sigma}_\mu)^{\dot{\gamma}\sigma} = 2\,\delta_\alpha^\sigma\delta_{\dot{\beta}}^{\dot{\gamma}}\,,$$
$$(\sigma^\mu)_{\alpha\dot{\beta}}\,(\sigma_\mu)_{\gamma\dot{\sigma}} = -2\,\epsilon_{\alpha\gamma}\bar{\epsilon}_{\dot{\beta}\dot{\sigma}}\,,$$
$$(\sigma^\mu)_{\sigma\dot{\delta}} = \bar{\epsilon}_{\dot{\delta}\dot{\alpha}}\,(\bar{\sigma}^\mu)^{\dot{\alpha}\beta}\,\epsilon_{\beta\sigma}\,,$$
$$(\bar{\sigma}^\mu)^{\dot{\alpha}\beta} = \epsilon^{\beta\sigma}\,(\sigma^\mu)_{\sigma\dot{\delta}}\,\bar{\epsilon}^{\dot{\delta}\dot{\alpha}}\,.$$

Show then that if λ and χ are two anticommuting two-component spinors (field operators or Grassmann variables) and $\bar{\lambda}$ and $\bar{\chi}$ are their conjugates, then

$$\bar{\chi}\bar{\sigma}^\mu\lambda = -\lambda\sigma^\mu\bar{\chi}\,,$$
$$(\chi\sigma^\mu\bar{\lambda})^\dagger = (\lambda\sigma^\mu\bar{\chi})\,.$$

Finally check if T^a_{ij} are some Hermitian matrices (e.g. matrix generators of an internal symmetry group),

$$\bar{\chi}_i\bar{\sigma}^\mu T^a_{ij}\lambda_j = \lambda_i\sigma^\mu(-T^{a*})_{ij}\bar{\chi}_j\,.$$

Problem 3.17
Check that the matrices M and $(M^\dagger)^{-1}$

$$M \equiv \exp\left(-\frac{i}{2}\,\sigma^k(\eta^k - i\xi^k)\right),$$
$$(M^\dagger)^{-1} \equiv \exp\left(-\frac{i}{2}\,\sigma^k(\eta^k + i\xi^k)\right),$$

of the two inequivalent $SL(2,C)$ representations of the lowest dimension can be written in the following covariant forms

$$M = \exp\left(-\frac{i}{2}\,\omega_{\mu\nu}\,\frac{1}{2}\sigma^{\mu\nu}_{2\times2}\right),$$
$$(M^\dagger)^{-1} = \exp\left(-\frac{i}{2}\,\omega_{\mu\nu}\,\frac{1}{2}\bar{\sigma}^{\mu\nu}_{2\times2}\right),$$

where

$$(\sigma^{\mu\nu}_{2\times2})_\alpha^{\ \beta} = \frac{i}{2}(\sigma^\mu\bar{\sigma}^\nu - \sigma^\nu\bar{\sigma}^\mu)_\alpha^{\ \beta}\,,$$
$$(\bar{\sigma}^{\mu\nu}_{2\times2})^{\dot{\alpha}}_{\ \dot{\beta}} = \frac{i}{2}(\bar{\sigma}^\mu\sigma^\nu - \bar{\sigma}^\nu\sigma^\mu)^{\dot{\alpha}}_{\ \dot{\beta}}\,.$$

Find how the parameters η^i and ξ^i are related to the parameters $\omega_{\mu\nu}$. Considering infinitesimal transformations, check by explicit calculation that for $\omega_{\mu\nu}$ related to the Lorentz transformation Λ by $\Lambda^\mu_{\ \nu} = \delta^\mu_{\ \nu} + \omega^\mu_{\ \nu}$ the following relations (expressing the "canonical" twofold covering of the Lorentz group by $SL(2C)$)

$$M(V_\mu\sigma^\mu)M^\dagger = \sigma^\mu\,V'_\mu\,,$$
$$(M^\dagger)^{-1}(V_\mu\bar{\sigma}^\mu)M^{-1} = \bar{\sigma}^\mu\,V'_\mu\,,$$

where $V'^\mu = \Lambda^\mu_{\ \nu}V^\nu$, which equivalently can be written in the form

$$\Lambda^\mu_{\ \nu}M\sigma^\nu M^\dagger = \sigma^\mu\,,$$
$$\Lambda^\mu_{\ \nu}(M^\dagger)^{-1}\bar{\sigma}^\nu M^{-1} = \bar{\sigma}^\mu\,,$$

are satisfied.[16] These relations imply that if V^μ is a Lorentz four-vector and λ and $\bar\chi$ transform as $(\frac{1}{2},0)$ and $(0,\frac{1}{2})$ representations of $SL(2,C)$, the quantities

$$V^\mu(\lambda\sigma_\mu\bar\chi)\,, \qquad \text{and} \qquad V^\mu(\bar\chi\bar\sigma_\mu\lambda)\,,$$

transform as scalars.

Problem 3.18
Check the following identities

$$\begin{aligned}
\mathrm{tr}(\sigma^\mu\bar\sigma^\nu\sigma^\lambda\bar\sigma^\kappa) &= 2\left(g^{\mu\nu}g^{\lambda\kappa} + g^{\mu\kappa}g^{\nu\lambda} - g^{\mu\lambda}g^{\nu\kappa}\right) + 2i\,\epsilon^{\mu\nu\lambda\kappa}\,,\\
\mathrm{tr}(\bar\sigma^\mu\sigma^\nu\bar\sigma^\lambda\sigma^\kappa) &= 2\left(g^{\mu\nu}g^{\lambda\kappa} + g^{\mu\kappa}g^{\nu\lambda} - g^{\mu\lambda}g^{\nu\kappa}\right) - 2i\,\epsilon^{\mu\nu\lambda\kappa}\,,
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{tr}\left(\sigma^{\mu\nu}_{2\times2}\,\sigma^{\lambda\kappa}_{2\times2}\right) &= 2\left(g^{\mu\lambda}g^{\nu\kappa} - g^{\mu\kappa}g^{\nu\lambda}\right) - 2i\,\epsilon^{\mu\nu\lambda\kappa}\,,\\
\mathrm{tr}\left(\bar\sigma^{\mu\nu}_{2\times2}\,\bar\sigma^{\lambda\kappa}_{2\times2}\right) &= 2\left(g^{\mu\lambda}g^{\nu\kappa} - g^{\mu\kappa}g^{\nu\lambda}\right) + 2i\,\epsilon^{\mu\nu\lambda\kappa}\,,
\end{aligned}$$

($\sigma^{\mu\nu}_{2\times2}$ and $\bar\sigma^{\mu\nu}_{2\times2}$ are defined in Problem 3.17) and

$$\begin{aligned}
\sigma^\mu\bar\sigma^\kappa\sigma^\lambda + \sigma^\lambda\bar\sigma^\kappa\sigma^\mu &= 2\left(g^{\mu\kappa}\sigma^\lambda + g^{\lambda\kappa}\sigma^\mu - g^{\mu\lambda}\sigma^\kappa\right),\\
\bar\sigma^\mu\sigma^\kappa\bar\sigma^\lambda + \bar\sigma^\lambda\sigma^\kappa\bar\sigma^\mu &= 2\left(g^{\mu\kappa}\bar\sigma^\lambda + g^{\lambda\kappa}\bar\sigma^\mu - g^{\mu\lambda}\bar\sigma^\kappa\right),
\end{aligned}$$

and, finally, that

$$\begin{aligned}
\sigma^\mu\bar\sigma^\kappa\sigma^\lambda - \sigma^\lambda\bar\sigma^\kappa\sigma^\mu &= \;\;2i\,\epsilon^{\mu\kappa\lambda\nu}\sigma_\nu\,,\\
\bar\sigma^\mu\sigma^\kappa\bar\sigma^\lambda - \bar\sigma^\lambda\sigma^\kappa\bar\sigma^\mu &= -2i\,\epsilon^{\mu\kappa\lambda\nu}\bar\sigma_\nu\,.
\end{aligned}$$

(Recall that we use $\epsilon^{0123}=+1$).

[16] For infinitesimal parameters $\omega_{\mu\nu}$ there is no ambiguity related to the fact the both M and $-M$ satisfy the same "canonical" definition.