Problem 5.1
Consider the scattering process $A+B \to C+D$. Show that in the center of mass system (CMS) the factor $F = 4\sqrt{(k_1 \cdot k_2)^2 - m_1^2 m_2^2}$ can be written as

$$F = 4|\mathbf{k}_i|\sqrt{s}\,,$$

where $\mathbf{k}_i = \mathbf{k}_A = -\mathbf{k}_B$ and

$$s = (k_A + k_B)^2 = (E_A + E_B)^2\,,$$

whereas the final state phase space factor $dQ = (2\pi)^4 \delta^{(4)}(k_A + k_B - p_C - p_D) d\Gamma_{\mathbf{p}_C} d\Gamma_{\mathbf{p}_D}$ in the expression $d\sigma = (1/F) \sum |\mathcal{A}|^2 dQ$ can be integrated to give

$$dQ = \frac{|\mathbf{p}_f|}{16\pi^2\sqrt{s}}\, d\Omega_{\mathbf{p}_f}\,,$$

where $\mathbf{p}_f = \mathbf{p}_C = -\mathbf{p}_D$ and $d\Omega_f = d\phi_C d\theta_C \sin\theta_C$ (ϕ_C and θ_C specify the direction of $\mathbf{p}_C$ with respect to $\mathbf{k}_A$), so that the differential cross section reads

$$d\sigma(\theta, \phi) = \frac{1}{64\pi^2 s}\,\frac{|\mathbf{p}_f|}{|\mathbf{k}_i|}\,|\mathcal{A}|^2\, d\Omega_{\mathbf{p}_f}\,.$$

Express $|\mathbf{k}_i|$ and $|\mathbf{p}_f|$ in terms of s and the particle masses.

Problem 5.2
The $pp \to \pi^+ D$ cross section (D stands for Deuterium of mass $M_D = 1874.98$ MeV) measured in the Hydrogen fixed target experiment with the proton kinetic energy[27] $T_p = 340$ MeV is $\sigma(pp \to \pi^+ D) = 0.18$ mb. In turn, the cross section $\sigma(\pi^+ D \to pp)$ measured in the Deuterium fixed target experiment with $T_\pi = 25$ MeV is about 3 mb. By appealing to the time reversal invariance of the strong interactions show that these result imply that pion is a spinless particle.

Problem 5.3
Using the weak interaction Hamiltonian

$$\mathcal{H}_{\rm weak} = \frac{G_F}{\sqrt{2}}\, J^\dagger_\lambda J^\lambda \qquad \text{where} \qquad J^\lambda = J^\lambda_{\rm lept} + J^\lambda_{\rm hadr}\,,$$

$$J^\lambda_{\rm lept} = \bar{\psi}_{(e)}\gamma^\lambda(I - \gamma^5)\psi_{(\nu_e)} + \bar{\psi}_{(\mu)}\gamma^\lambda(I - \gamma^5)\psi_{(\nu_\mu)} + \bar{\psi}_{(\tau)}\gamma^\lambda(I - \gamma^5)\psi_{(\nu_\tau)}\,,$$

compute the differential (with respect to the final charged lepton direction) and the total cross sections of the processes $\nu_\mu e^- \to \nu_e \mu^-$ and $\bar{\nu}_e e^- \to \bar{\nu}_\ell \ell^-$. Perform calculations both in the CMS and in the Laboratory system (electron initially at rest). Give the total CMS cross sections in barns ($1b = 10^{-28}\ m^2$) for $\sqrt{s} = 10$ MeV and 100 GeV. What is the minimal (threshold) energy of ν_μ capable to initiate the process $\nu_\mu e^- \to \nu_e \mu^-$ in the

[27] By kinetic energy one means $T_p \equiv E_p - m_p c^2 = \sqrt{\mathbf{k}_p^2 c^2 + m_p^2 c^4} - m_p c^2$.

Laboratory system? Explain the angular dependence of these differential cross sections in the limit in which lepton masses can be neglected by appealing to angular momentum conservation. What is the cross section for the process $\bar{\nu}_\mu e^- \to \bar{\nu}_e \mu^-$?

Problem 5.4
Using the Hamiltonian given in Problem 5.3 find the partial wave amplitudes $\mathcal{T}^{(j)}_{\lambda_\ell \lambda_{\nu_e}, \lambda_{\nu_\ell} \lambda_e}(s)$ of the process $\nu_\ell e^- \to \nu_e \ell^-$ and determine the energy at which the lowest order (in G_F) elastic scattering amplitude fails to satisfy the unitarity bound. Ignore the possible existence of the neutral currents interaction (that is, mediated by the Z^0 boson).

Problem 5.5
Assume that the (charged currents induced) weak interactions are mediated by the spin 1 particle $W^\pm$ of mass $M_W \gg m_e$, so that the Hamiltonian of weak leptonic processes is

$$\mathcal{H}_{\text{weak}} = \frac{g_2}{2\sqrt{2}} J^\lambda W^-_\lambda + \frac{g_2}{2\sqrt{2}} (J^\lambda)^\dagger W^+_\lambda \,.$$

Find the partial wave amplitudes of the process $\nu_\ell e^- \to \nu_e \ell^-$ and reconsider the determination of the unitarity bound.

Problem 5.6
Consider a field theory of four real scalar fields π^a, $a = 1, 2, 3$ and η with the Lagrangian[28]

$$\mathcal{L} = \frac{1}{2} \sum_{a=1}^{3} \left(\partial_\mu \pi^a \partial^\mu \pi^a - M_\pi^2 \, \pi^a \pi^a \right) + \frac{1}{2} \partial_\mu \eta \partial^\mu \eta - \frac{1}{2} M_\eta^2 \eta^2$$
$$- \frac{\kappa}{2} \left(\eta^2 + \sum_{a=1}^{3} \pi^a \pi^a \right) \eta - \frac{\lambda}{4} \left(\eta^2 + \sum_a \pi^a \pi^a \right)^2 .$$

Find in the lowest order amplitudes of the processes $\pi^+\pi^- \to \pi^0\pi^0$, $\pi^+\pi^+ \to \pi^+\pi^+$, $\pi^+\pi^0 \to \pi^+\pi^0$ etc. where the one particle states of π^+, π^- and π^0 are the common eigenstates of H_0, $\hat{\mathbf{T}}^2 \equiv (\hat{T}^1)^2 + (\hat{T}^2)^2 + (\hat{T}^3)^2$ (the total isospin) and $\hat{T}^3$ (the isospin third component) operators found in Problem **??**. Construct the S-matrix elements in the isospin basis

$$S_{I', I'_3; I, I_3} = \langle I', I'_3, \mathbf{p}_1, \mathbf{p}_2 | T \exp\left(i \int d^4x \, \mathcal{L}_I \right) | I, I_3, \mathbf{k}_1, \mathbf{k}_2 \rangle \,,$$

where $|I, I_3, \mathbf{k}_1, \mathbf{k}_2\rangle$ are the two-particle eigenstates of H_0, $\hat{\mathbf{T}}^2$ and of $\hat{T}^3$. Check by direct computation that $S_{I', I'_3; I, I_3} = S^I \delta_{I', I} \delta_{I'_3, I_3}$ that is, that the amplitudes do not depend on I_3.

[28] Equivalently, one can consider the interaction Hamiltonian density

$$\mathcal{H}_{\text{int}}(x) = \frac{\kappa}{2} \left(\eta^2 + \sum_{a=1}^{3} \pi^a \pi^a \right) \eta + \frac{\lambda}{4} \left(\eta^2 + \sum_a \pi^a \pi^a \right)^2 ,$$

identifying the field operator associated with the π^0 particle with π^3 and those associated with the particle π^- and with its antiparticle π^+ with $(\pi^1 \pm i\pi^2)/\sqrt{2}$, respectively.

Express the amplitudes $\mathcal{A}$ of all possible $\pi\pi$ scatterings in terms of the isospin amplitudes $\mathcal{A}^I$.

By considering transitions between all possible pairs of two-particle states (including the η particle) show that in the limit $\sqrt{s} \gg M_\eta > M_\pi$, where $s = (k_1 + k_2)^2$, there are only three independent nonzero amplitudes which correspond to diagonal transitions within three different representations of the $SO(4)$ group realizable on two-particle states of spinless particles.

Problem 5.7

A realistic effective theory of interactions of low energy pions (in the limit of vanishing their masses) has the Lagrangian density[29]

$$\mathcal{L} = \frac{f_\pi^2}{4}\mathrm{tr}\left(\partial^\mu U \partial_\mu U^{-1}\right) + \dots\,,$$

(the ellipses stand for terms with more derivatives) where $U^{-1} = \exp\left(i\tau^a\pi^a/f_\pi\right)$ with τ^a the three Pauli matrices and $f_\pi \approx 93$ MeV called the pion decay constant (its value is determined in Problem VI.9). Using this Lagrangian find in the lowest order the amplitudes of all possible binary scatterings ($\pi^+\pi^+ \to \pi^+\pi^+$, $\pi^+\pi^- \to \pi^+\pi^-$, $\pi^+\pi^- \to \pi^0\pi^0$, etc.) and show as in Problem 5.6 that

$$\mathcal{A}(I, I_3, \mathbf{k}_1, \mathbf{k}_2 \to I', I'_3, \mathbf{p}_1, \mathbf{p}_2) = \delta_{II'}\,\delta_{I_3 I'_3}\mathcal{A}^I(\mathbf{k}_1, \mathbf{k}_2 \to \mathbf{p}_1, \mathbf{p}_2)\,.$$

Find the isospin amplitudes $\mathcal{A}^I \equiv \mathcal{A}(I, I_3, \mathbf{k}_1, \mathbf{k}_2 \to I, I_3, \mathbf{p}_1, \mathbf{p}_2)$.

Problem 5.8

The (fictitious) Hamiltonian of the three π mesons of masses M_π interacting with a neutral spinless particle η of mass M_η has the form

$$\mathcal{H}_{\rm int}(x) = \frac{\kappa}{2}\left(\eta^2 + \sum_{a=1}^{3}\pi^a\pi^a\right)\eta + \frac{\lambda}{4}\left(\eta^2 + \sum_{a=1}^{3}\pi^a\pi^a\right)^2.$$

Find the partial amplitudes $\mathcal{T}^{(l)}(s)$ of the elastic $\pi^+\pi^-$ scattering defined by the expansion of the scattering amplitude $\mathcal{A}$

$$\mathcal{A}(s, \cos\theta) = 16\pi\sum_{l=0}^{\infty}(2l+1)\mathcal{T}^{(l)}(s)P_l(\cos\theta)\,,$$

where $P_l(x)$ are the Legendre polynomials. Express the differential and total cross sections in the CMS system through the amplitudes $\mathcal{T}^{(l)}(s)$. What constraints on the coupling

[29] Again, for the interaction term $\mathcal{H}_{\rm int}$ generating Feynman rules one should take $-\mathcal{L}'$, where

$$\mathcal{L}' = \mathcal{L} - \frac{1}{2}\sum_{a=1}^{3}\partial_\mu\pi^a\partial^\mu\pi^a\,.$$

constants λ and κ follow from the (asymptotic) unitarity bounds

$$N\left|\mathcal{T}^{(l)}(s)\right| < 1\,, \qquad N\left|\mathrm{Re}\,\mathcal{T}^{(l)}(s)\right| < \frac{1}{2}\ ?$$

($N = 1$ for different particles and $N = \frac{1}{2}$ for identical final state particles). Optimize the constraint on λ by considering amplitudes of all possible binary reactions (including also those involving the η particle) for $\sqrt{s} \gg M_\eta,\ M_\pi$.

Observe, that the partial wave amplitude $\mathcal{T}^{(l=0)}(s)$ of the elastic $\pi^+\pi^-$ scattering computed in the lowest order has a simple pole at $s = M_\eta^2$. Check that including the width of the η particle in its tree level propagator by the substitution

$$\frac{i}{q^2 - M_\eta^2} \qquad \to \qquad \frac{i}{q^2 - M_\eta^2 + iM_\eta\Gamma_\eta}\,,$$

(with Γ_η computed in the lowest order) restores unitarity of the scattering amplitude saturating (up to nonresonant terms) the basic unitarity relation (for $l = 0$).

Problem 5.9
Consider the (fictitious) interaction of the charged particles π of mass M_π with a massive spin one particle of mass M_V

$$\mathcal{L}_{\mathrm{int}} = -igV^\mu\left(\pi^*\partial_\mu\pi - \partial_\mu\pi^*\pi\right) - \lambda(\pi^*\pi)^2\,.$$

Find the partial wave amplitudes $\mathcal{T}^{(l)}(s)$ of the elastic $\pi^+\pi^-$ scattering. As in the preceding problem investigate the constraints imposed by unitarity on the partial amplitudes $\mathcal{T}^{(l)}(s)$.

Problem 5.10
Imposing the unitarity bounds on the pion scattering amplitudes determine the range of energies for which the interaction (see Problem 5.7)

$$\mathcal{L} = \frac{f_\pi^2}{4}\mathrm{tr}\left(\partial_\mu U\partial_\mu U^{-1}\right) + \dots\,,$$

can be used in the tree level approximation.

Problem 5.11
Find the amplitudes of binary pion scatterings as in Problem 5.7 but taking into account finite pion masses by using the Lagrangian[30]

$$\mathcal{L} = \frac{f_\pi^2}{4}\,\mathrm{tr}\left(\partial_\mu U\partial_\mu U^{-1}\right) + \frac{f_\pi^2 M_\pi^2}{4}\,\mathrm{tr}\left(U + U^{-1}\right) + \dots\,,$$

In the lowest order find the pion scattering phase shifts $\delta_I^{(l)}$ and the pion scattering lengths.

[30] The interaction $\mathcal{H}$ shuld be constructed by subtracting from the expanded form of $\mathcal{L}$ terms quadratic in the fields π^a and taking minus of the rest.

Problem 5.12
Taking the interaction in the form

$$\mathcal{H} = ig_{\pi N}\, \pi^a \bar{\psi}_N \tau^a \gamma^5 \psi_N\,,$$

where π^a shoud be treated as in Problem 5.6, τ^a are the three Pauli matrices and ψ_N is a doublet composed of the nucleon fields ψ_p and ψ_n (ψ_p as its isospin $I_3 = +1/2$ component and ψ_n as the $I_3 = -1/2$ one) write down the lowest order (in the coupling constant $g_{\pi N}$) amplitudes of the pion-nucleon scattering. Check by direct calculation that $S_{I',I'_3;I,I_3} = S^I \delta_{I',I}\delta_{I'_3,I_3}$.

Problem 5.13
Consider the Yukawa interaction $\mathcal{H}_{\rm int} = h\,\bar{\psi}\psi\varphi$, where h is the coupling constant, of a spinless neutral particle represented by the operator η with fermions f (and their antifermions $\bar{f}$) represented by the operators ψ and $\bar{\psi}$. Compute in the lowest order in h the differential cross section of the elastic scattering $\eta\bar{f} \to \eta\bar{f}$. Assume that the initial antifermions are unpolarized and the final antifermion spin is not measured.

Problem 5.14
Let ψ_a and ψ_b be the field operators of fermions (antifermions) f_a ($\bar{f}_a$) and f_b ($\bar{f}_b$) with masses m_a and m_b, respectively. Using the interaction $\mathcal{L}_{\rm int} = -ig_a\bar{\psi}_a\gamma^5\psi_a\varphi - ig_b\bar{\psi}_b\gamma^5\psi_b\varphi$ compute in the CMS the differential and total cross sections for processes: $f_a\bar{f}_a \to f_b\bar{f}_b$, $f_a\bar{f}_b \to f_a\bar{f}_b$ and $f_a\bar{f}_a \to f_a\bar{f}_a$. Assuming that m_a, $m_b \ll M$, where M is the mass of the neutral spinless particle described by φ write down in each case the effective Lagrangian with contact interaction of fermions which reproduces the scattering amplitude for energies of the colliding fermions much smaller than M.

Problem 5.15
Using the standard interactions of electrons, positrons, muons and antimuons with the photons compute (in the lowest order in e) the CMS differential and total cross sections for production of a $\mu^-\mu^+$ pair in the e^-e^+ collision. Compare the angular distribution of the produced μ^- with the distribution of (hypotetical) spinless $\tilde{\mu}$ particles produced in the e^-e^+ collision.

Problem 5.16
Explain the dependence on the scattering angle of the cross sections computed in Problem 5.15 by studing these processes of annihilation and production of particles with definite helicities in the high energy limit (in which particle masses can be neglected).

Problem 5.17
Using the covariant Feynman rules of quantum electrodynamics of spin 1/2 charged particles (and their antiparticles) write down the lowest order (in the coupling e) amplitude of the elastic $\gamma\, e^-$ scattering (the Compton process). Check that the amplitude is gauge invariant, that is, vanishes when any of the two photon polarization vectors, $\epsilon_\mu(\mathbf{k}_1, \lambda_1)$ or $\epsilon^*_\nu(\mathbf{k}_2, \lambda_2)$, is replaced by the four-momentum, k_1^μ or k_2^ν, of the corresponding photon. Assuming that the initial electron is at rest compute the differential and total cross sections

as functions of the polarizations of the initial and final photons. Find the low energy limit ($|\mathbf{k}_1| \to 0$) of the these cross sections.

Problem 5.18

Supersymmetric theories predict the existence of a spin 0 partner of each fermion (e.g. the supersymmetric partners of $e^{\pm}$ are the *selectrons* $\tilde{e}^{\pm}$) and of the neutral fermions N^0 called neutralinos (which are supersymmetric partners of the Higgs and gauge bosons). Calculate (in the lowest order in the relevant coupling constants) the differential cross section of the process $\gamma N^0 \to e^- \tilde{e}^+$. Assume the most general (not necessarily parity conserving) form of the neutralino-electron-selectron interaction vertex (photon interaction vertices are standard). Fix the relative sign between the two amplitudes contributing in the lowest order by appealing to the gauge invariance of the amplitude.

Problem 5.19

The couplings of mass M spinless particles of electric charge Q (in units $e > 0$) with with photons have the form

$$\mathcal{H}_{\rm int} = -ieQA^{\mu}(\partial_{\mu}\phi^{\dagger}\phi - \phi^{\dagger}\partial_{\mu}\phi) - e^2Q^2A^{\mu}A_{\mu}\phi^{\dagger}\phi\,.$$

Consider scattering of a photon on such a spinless particle in the Laboratory frame (in which the spinless particle is initially at rest). Compute in the lowest order approximation the differential cross section for finding the scattered photon at an angle θ (with respect to the direction of the initial photon) with polarization λ_2, if the initial photon has four-momentum $\mathbf{k}_1$ and polarization λ_1. Find also the differential cross section averaged over polarizations of the initial photon in the case the polarization of the final photon is not measured. To compute the latter cross section, construct explicitly the polarization vectors of the photons choosing them to be purely spatial (this eliminates two of the three terms in the amplitude) and perform the necessary sumation over polarizations using these vectors. To appreciate, how much more efficient this approach is, recover the same result using the Feynman trick $\sum_{\lambda} \epsilon_{\mu}(\mathbf{k}, \lambda)\epsilon^*_{\nu}(\mathbf{k}, \lambda) \to -g^{\mu\nu}$.

Problem 5.20 (Numerical Exercise)

Consider the production process $S_1(\mathbf{k}_1)+S_2(\mathbf{k}_2) \to \tilde{S}_1(\mathbf{p}_1)+\tilde{S}_2(\mathbf{p}_2)+\tilde{S}_3(\mathbf{p}_3)$ where S_i and $\tilde{S}_i$ are all massive spinless particles. The proces occurs (in the lowest order) through the s-channel annihilation of S_1S_2 into a (virtual) spinless particle of mass m which goes into $\tilde{S}_3$ and another spinless particle S^* of mass M and width $\Gamma_{\rm tot}$ which decays producing $\tilde{S}_2$ and $\tilde{S}_1$ (there may also be other Feynman diagrams contributing to the total amplitude $\mathcal{A}[S_1(\mathbf{k}_1) + S_2(\mathbf{k}_2) \to \tilde{S}_1(\mathbf{p}_1) + \tilde{S}_2(\mathbf{p}_2) + \tilde{S}_3(\mathbf{p}_3)]$ which need not be taken into account). Assuming that the relevant interactions have the forms of products $\kappa\varphi_1\varphi_1\varphi_3$ (with some couplings κ) of three scalar operators associated with the spin 0 particles involved, show that if $\Gamma_{\rm tot} \ll M$ (S^* is a narrow width resonance) then for $\sqrt{s} > M$ the cross section $\sigma(S_1S_2 \to \tilde{S}_1\tilde{S}_2\tilde{S}_3)$ can be approximated by

$$\sigma(S_1S_2 \to \tilde{S}_1\tilde{S}_2\tilde{S}_3) \approx \sigma(S_1S_2 \to \tilde{S}_3S^*) \times {\rm Br}(S^* \to \tilde{S}_1\tilde{S}_2)\,.$$

Compare this approximation with the full $\sigma(S_1S_2 \to \tilde{S}_1\tilde{S}_2\tilde{S}_3)$ cross section numerically by taking the initial and final particles to be massless (so that the results of the Problem V.2

for the final phase space can be used). Take e.g. $m = 100$ GeV, $M = 10$ GeV (so that the peak associated with the s-channel resonanse of mass m does not distort the cross section appreciably) and plot both cross sections as a function of $\sqrt{s}$ for 1 GeV $< \sqrt{s} <$ 30 GeV and several values of $\Gamma_{\rm tot}$.